On the Structure of Unconditional UC Hybrid Protocols

Mike Rosulek[1⋆] and Morgan Shirley[2]

[1] Oregon State University, rosulekm@eecs.oregonstate.edu
[2] University of Toronto, shirley@cs.toronto.edu

Abstract. We study the problem of secure two-party computation in the presence of a trusted setup. If there is an unconditionally UC-secure protocol for f that makes use of calls to an ideal g, then we say that f *reduces to* g (and write $f \sqsubseteq g$). Some g are *complete* in the sense that *all* functions reduce to g. However, almost nothing is known about the power of an incomplete g in this setting. We shed light on this gap by showing a characterization of $f \sqsubseteq g$ for incomplete g.

Very roughly speaking, we show that f reduces to g if and only if it does so by the simplest possible protocol: one that makes a single call to ideal g and uses no further communication. Furthermore, such simple protocols can be characterized by a natural combinatorial condition on f and g.

Looking more closely, our characterization applies only to a very wide class of f, and only for protocols that are deterministic or logarithmic-round. However, we give concrete examples showing that both of these limitations are inherent to the characterization itself. Functions not covered by our characterization exhibit qualitatively different properties. Likewise, randomized, superlogarithmic-round protocols are qualitatively more powerful than deterministic or logarithmic-round ones.

1 Introduction

In 2-party secure function evaluation (SFE), Alice holds a private input $x \in X$, Bob holds a private input $y \in Y$, and the parties interact to learn $f(x, y)$ for some agreed-upon function $f : X \times Y \to Z$. Each party should learn no more than can be inferred from $f(x, y)$ alone, even when that party behaves adversarially.

Different functions f have different inherent complexities, and one way to compare the "cryptographic complexity" of functions is to use a *reduction.* The most natural reduction from f to g is a secure protocol for f where the parties are allowed to use an ideally-secure black-box for g (ideal here means that this black-box takes inputs from both parties, and reveals only the output of g). Depending on the security required of the protocol for f, we obtain reductions of various strengths that can resolve finer distinctions in cryptographic complexity.

⋆ Partially supported by NSF awards #114964 & #1617197.

Cryptographic complexity & related work. In this work we exclusively focus on reductions between two-party, deterministic SFE functions with constant-size truth tables (meaning the function that is computed does not depend on the security parameter). We consider a reduction defined in terms of UC security [4] against computationally unbounded adversaries. We write $f \sqsubseteq g$ to denote that there is a UC-secure protocol securely realizing f against unbounded adversaries, that makes calls to an ideal g functionality (*i.e.*, a protocol in the "g-hybrid model").

After defining a notion of reducibility, the most natural step is to identify which objects are *complete* for the reduction. A function g is **complete** (under $\sqsubseteq$) if $f \sqsubseteq g$ for all f. Otherwise we say that g is **incomplete**.

Kilian [8] was the first to consider completeness of SFE functionalities, proving that the oblivious transfer function is complete. Although the result predates the UC model, a variant of the construction from [9] is likely to achieve UC security — *i.e.*, oblivious transfer is complete under the $\sqsubseteq$ reduction that we consider in this work. Later work characterized exactly which functions are complete (w.r.t. malicious, unconditional security): for symmetric SFE (where both parties receive the same output) [10], for asymmetric SFE (where only one party receives output) [11], for SFE where parties may receive different outputs [13], and even for randomized SFE functions [20,12].

When one or both of f, g are complete, the question of $f \sqsubseteq g$ is simple to answer. If g is complete, then $f \sqsubseteq g$. If f is complete but g is not, then $f \not\sqsubseteq g$. The goal of this line of work is to therefore **understand when $f \sqsubseteq g$, for f and g which are incomplete.**

Prabhakaran & Rosulek [21] gave an example of four functions that satisfy $f_1 \sqsubset f_2 \sqsubset f_3 \sqsubset f_4$ (where $f \sqsubset g$ means that $f \sqsubseteq g$ but $g \not\sqsubseteq f$). Maji, Prabhakaran & Rosulek [17] extended this result to show an *infinite* strict hierarchy $f_1 \sqsubset f_2 \sqsubset \cdots \sqsubset f_i \sqsubset \cdots$, and also showed an example of a pair of functions that are incomparable ($f \not\sqsubseteq g$ and $g \not\sqsubseteq f$). The same authors in [18] later proved several additional results of the form $f \not\sqsubseteq g$.

These results hint at a rich landscape of complexity with respect to the $\sqsubseteq$ reduction, but fall well short of revealing the entire picture. First, they are not complete characterizations, but give only necessary conditions for $f \sqsubseteq g$. Second, the techniques in these works apply only to f and g that have *semi-honest-secure* protocols. This leaves a large class of functions that are neither complete nor admit any semi-honest protocol (simple characterizations of both properties are known [1,14,10]). A canonical example of such a function is the so-called "spiral" function shown in Figure 1c. Currently almost nothing is known about reductions involving such intermediate functions.

Other related work. We consider a reduction based on UC security against unbounded adversaries. Weaker reductions have been studied, but they do not turn out to illuminate many distinctions in complexity. For example, one may define a reduction based on *polynomial-time* UC security. It turns out that every SFE is either complete or trivial (it reduces to every other function) under this reduction [19].

In this paper we prove qualitative differences in the power of randomized and deterministic protocols, even when realizing deterministic functions using other deterministic functions (*i.e.*, we show specific, deterministic f and g where $f \sqsubseteq g$ via a randomized protocol but not by any deterministic protocol). In the two-party setting, Dodis & Micali [5] show such a separation among *complete* f and g, for a special class of protocols (in which one party does not speak). Beimel & Malkin [2] show specific f and (complete) g for which randomized protocols make exponentially fewer calls to g than deterministic protocols.

2 Overview of Our Results

Scope of Results: Incomplete, Non-Unilateral Functions. As mentioned in the introduction, the question of $f \sqsubseteq g$ is straight-forward when one of $\{f, g\}$ are complete. We therefore focus on characterizing $f \sqsubseteq g$ when both are incomplete.

We say that f is **unilateral** if there exists an input y^* for one of the parties (by symmetry, Bob) such that $f(\cdot, y^*)$ is a constant function. That is, by choosing input y^* Bob can *unilaterally* fix the output of f. We characterize $f \sqsubseteq g$, when f is non-unilateral. In Section 7.1 we show an example of unilateral f, g that do *not* obey the characterization, demonstrating that this restriction is tight.

Statement of Main Result. We show a complete and combinatorial characterization of $f \sqsubseteq g$, for a natural class of protocols. For this characterization, identify each function f with its 2-dimensional truth table (rows corresponding to Alice-inputs and columns to Bob-inputs). We say that f **embeds in** g if f appears as a submatrix of g, subject to some other restrictions (essentially, the other parts of g can't "interfere" with the f-submatrix — the formal definition is in Section 4). We then prove our main theorem:

Theorem 1. *The following are equivalent, when f and g are incomplete and f is non-unilateral.*

1. *$f \sqsubseteq g$ via a worst-case $O(\log \kappa)$-round protocol (where κ is the security parameter)*
2. *$f \sqsubseteq g$ via a deterministic protocol*
3. *$f \sqsubseteq g$ via a deterministic protocol consisting of a single call to g and no additional communication.*
4. *f embeds in g*

Technical Approach. The most involved part of our main theorem is proving (1) $\Rightarrow$ (3) and (2) $\Rightarrow$ (3). Intuitively this involves "compressing" an arbitrary protocol for $f \sqsubseteq g$ into a single call to g.

Our first step is to show that every secure protocol for $f \sqsubseteq g$ can be transformed into one with the following **instantaneous** property:

- With overwhelming probability, the protocol terminates immediately following some call to g.
- Strictly before this terminal call to g, the protocol transcript leaks negligible information about either party's inputs.

Our main technical tool is that of *frontier analysis*, which was introduced in [17] and extended in [16]. A *frontier* in the protocol is simply the collection of partial transcripts where some statistical condition is true for the first time. Roughly speaking, we define two frontiers for each party: one expressing "the first time the simulator is likely to extract" and another for "the first time honest parties can reliably predict the final output." We then argue that these frontiers must all be reached simultaneously, with overwhelming probability. As such, these events can happen only as the result of a call to g. Furthermore, the protocol can be safely truncated after reaching the frontiers (since both parties can already predict the final output). The result of truncation is a protocol with the "instantaneous" property described above. We complete the argument by showing how such an instantaneous protocol can be compressed from $O(\log \kappa)$ rounds to one round.

Tightness of the Characterization Our main theorem does not characterize $f \sqsubseteq g$ when f is unilateral. This restriction is inherent, as we demonstrate with an example in Section 7.1. In Section 7.2 we also demonstrate an example f and g with the following properties:

1. f does not embed in g. Hence, by the classification theorem, $f \not\sqsubseteq g$ via any deterministic protocol or (randomized) logarithmic-round protocol.
2. $f \sqsubseteq g$ via a randomized protocol whose *expected* round complexity is constant, but whose *worst-case* round complexity is $r(\kappa)$ for *any* $r(\kappa) = \omega(\log \kappa)$.

This example demonstrates that our main characterization's limitation to $O(\log \kappa)$-round protocols is inherent.

Interestingly, the $\omega(\log \kappa)$-round protocol for $f \sqsubseteq g$ has the *instantaneous* property described above. Hence, the protocol leaks no information about the parties' inputs, until $f(x, y)$ is completely revealed in a single call to g. Yet there is no way to securely compress the protocol to just the "meaningful" call to g. Somehow, it is important that the "output-fixing" round is unpredictable.

Mysteriously, a similar structure appears in the protocols of Gordon *et al.* [6] that achieve fairness. These protocols leak nothing about the inputs until, in some secret round, the output is completely revealed. Analogously, Lindell & Rabin [15] show that the "output-fixing" round in a fair protocol cannot be predictable. We are not sure what fairness has to do with *unfair* multi-party computation with an incomplete hybrid functionality, and leave open this exploration for future work.

Ours is also one of the few examples of a $\omega(1)$ round-complexity lower bound for information-theoretic multi-party computation. Indeed, when g is complete, $f \sqsubseteq g$ is possible in constant rounds, for any f: first, obtain oblivious transfer from g in constant rounds [10,13], and from oblivious transfer obtain f in constant rounds [7].[3]

[3] Note that the round complexity of these protocols may depend on f and g (e.g., the circuit depth of f), but is constant with respect to the security parameter since we consider only functions with constant-size truth table.

3 Preliminaries

3.1 Secure Function Evaluation, UC Security

We assume the reader has familiarity with the UC framework (a brief overview is given in Appendix A). In this work we study deterministic 2-party secure function evaluation (SFE), in the universal composability (UC) framework [4] against computationally unbounded adversaries that corrupt parties *statically* (*i.e.*, once and for all before the protocol begins). We consider security-with-abort, meaning that malicious parties are allowed to learn their output first, and delay the honest parties from receiving output (perhaps indefinitely).

We use the following notation:

$f \sqsubseteq g$: there is a secure protocol (UC, unconditional) for f that uses calls to an ideal g (i.e., a secure protocol in the g-hybrid model).

$f \sqsubseteq_1 g$: $f \sqsubseteq g$ via a protocol that makes only a single call to g and uses no additional communication.

3.2 Combinatorial Properties of Complete/Incomplete f

We review some basics of 2-party SFE. Let f be a 2-party SFE with domain $X \times Y$. A fundamental property of SFE has to do with decomposing the function into "rectangles." Define $\mathsf{rect}_f(x,y) = \{x' \mid f(x,y) = f(x',y)\} \times \{y' \mid f(x,y) = f(x,y')\}$. We refer to $\mathsf{rect}_f(x,y)$ as a **rectangle** of f.

The characterization of [in]completeness for 2-party SFE is due to Kilian:

Theorem 2 ([10]). *f is **incomplete** if and only if: for all x, x', y, y',*

$$f(x,y) = f(x',y) = f(x,y') \implies f(x',y') = f(x,y).$$

A useful consequence of Theorem 2 is the following:

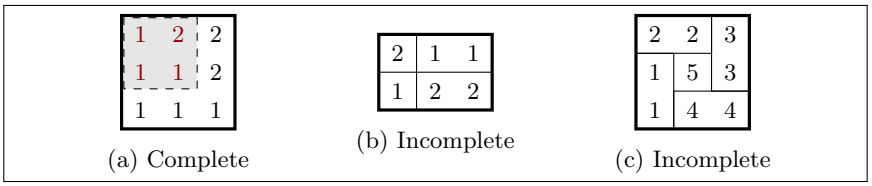
Observation 3 *Let f be incomplete. Then for all x, y, the value $\mathsf{rect}_f(x,y)$ is uniquely determined by $f(x,y)$ and just one of $\{x,y\}$. Likewise, $f(x,y)$ is uniquely determined by $\mathsf{rect}_f(x,y)$ and just one of $\{x,y\}$.*

Observation 3 implies that without loss of generality we can think of the parties as computing the function $(x,y) \mapsto \mathsf{rect}_f(x,y)$ instead of the function $f(x,y)$.

Figure 1 shows three example functions, with the partition into rectangles given for the incomplete functions. Note that the second function has 4 rectangles: two rectangles with output 1, and two with output 2.

In this work we restrict our attention to **symmetric functions**, which give the same output $f(x,y)$ to both parties. One could easily consider asymmetric functions $f = (f_A, f_B)$ which give output $f_A(x,y)$ to Alice and $f_B(x,y)$ to Bob. However, a result of Kraschewski & Müller-Quade [13] shows that all incomplete functions (even asymmetric ones) are isomorphic to some symmetric one.[4] Hence,

[4] If $f = (f_A, f_B)$ is incomplete then there is a symmetric function g such that (1) $g(x,y)$ can be computed from $x, f_A(x,y)$; (2) $g(x,y)$ can be computed from $y, f_B(x,y)$; (3) $f_A(x,y)$ can be computed from $x, g(x,y)$; (4) $f_B(x,y)$ can be computed from $y, g(x,y)$.



(a) Complete (b) Incomplete (c) Incomplete

Fig. 1. Example SFE functions.

our restriction to symmetric functions is without loss of generality, and our theorem statements can be interpreted to apply to asymmetric functions as well.

3.3 Properties of g-Hybrid Protocols for Incomplete g

Fix a 2-party protocol π, and let t be a partial transcript (*i.e.*, a prefix of a complete protocol transcript). We use $\underline{\Pr_\pi[t|xy]}$ to denote the probability of obtaining a protocol transcript with prefix t, when both parties run the protocol honestly with respective inputs x and y.

Write t as a sequence of messages $t = (m_1, \ldots, m_k)$. Suppose Alice sends the odd-numbered messages. Then the choice of the odd-numbered (resp. even-numbered) messages depends only on the previous messages and x (resp. y), but not on y (resp. x). We can therefore write:

$$\begin{aligned}\Pr_\pi[t|xy] &= \textstyle\prod_{i=1}^{k} \Pr_\pi[m_i|xy, m_1 \cdots m_{i-1}] \\ &= \Big(\textstyle\prod_{i \text{ odd}} \Pr_\pi[m_i|x, m_1 \cdots m_{i-1}]\Big)\Big(\textstyle\prod_{i \text{ even}} \Pr_\pi[m_i|y, m_1 \cdots m_{i-1}]\Big) \\ &\stackrel{\text{def}}{=} \Pr_\pi[t|x] \Pr_\pi[t|y] \qquad (\star)\end{aligned}$$

Here we are defining $\Pr_\pi[t|x]$ and $\Pr_\pi[t|y]$ to be equal to the parenthesized quantities. Essentially, $\Pr_\pi[t|x]$ is the probability that Alice behaves consistently with t when her input is x.

In the g-hybrid model. A similar property also holds when the parties can call an ideal functionality g (*i.e.*, protocols in the g-hybrid model), but only when g is *incomplete*, as we describe below.

When parties invoke g, its output is added to the joint transcript. From Observation 3 this is equivalent to adding $\mathsf{rect}_g(\tilde{x}, \tilde{y})$ to the transcript, where $\tilde{x}$ and $\tilde{y}$ were the inputs that the parties gave to this instance of g. Let $\tilde{X} \times \tilde{Y}$ be a particular rectangle in g. Then when g is incomplete we have:

$$\Pr[\mathsf{rect}_g(\tilde{x}, \tilde{y}) = \tilde{X} \times \tilde{Y}] = \Pr[\tilde{x} \in \tilde{X}] \Pr[\tilde{y} \in \tilde{Y}]$$

Alice's choice of $\tilde{x}$ depends only on her f-protocol input and the transcript so far; similarly $\tilde{y}$ depends only on Bob's input and the transcript so far. Hence, even though the parties contribute *simultaneously* to the transcript via a call to g (unlike when they alternate exchanging plain messages), the probability of a transcript can still be factored into *independent* contributions from the two parties, as in $(\star)$.

Stateless parties/adversaries. The "standard" way of defining a protocol is for each party to initially choose a random tape. Their subsequent behavior is a deterministic function of the random tape, their input, and the transcript so far.

However, ($\star$) shows that Alice's view (including her private randomness) is independent of Bob's view (including his randomness), given the transcript. Therefore, *any g-hybrid protocol π* can be purged of stateful randomness in the following way. At each step, a stateless party can (1) sample a random tape conditioned on it being consistent with their private input and transcript so far; (2) use that (ephemeral) random tape to choose the next move in the protocol; (3) discard the ephemeral random tape. Note that this transformation may require exponential time, but we consider all parties to have unbounded computation. This transformation *also applies to adversaries*, so without loss of generality we consider only stateless adversaries.

4 Reducibility Characterization

We define the combinatorial condition at the heart of our main theorem. Intuitively, f embeds to g if one can identify a submatrix of g that "looks like" f. Of course, the outputs of f might be renamed relative to g. Such a submatrix property suffices for a semi-honest protocol for f using g, where parties simply use the subset of inputs of g that comprise the f-submatrix. However, such a protocol need not be secure in the presence of *malicious* adversaries, because other inputs of g may "interfere" with the f-submatrix. There are two main things that can go wrong, epitomized in the following examples:

$$f_1 = \begin{bmatrix} 1 & 3 \\ 1 & 4 \\ 2 & 4 \end{bmatrix} \not\sqsubseteq \begin{bmatrix} 1 & 3 & 5 \\ 1 & 4 & 6 \\ 2 & 4 & 7 \end{bmatrix} = g_1; \qquad f_2 = \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix} \not\sqsubseteq \begin{bmatrix} 1 & 3 \to 3 \\ 2 \leftarrow 2 & 4 \end{bmatrix} = g_2$$

Note that f_1 appears as the white submatrix of g_1. However, when a corrupt column-player cheats and uses the shaded column of g_1, he completely learns the row-players input, even though no column of f_1 legally allows this.

Similarly, f_2 appears as a submatrix of g_2. Consider a corrupt column-player who uses the shaded column of g_2. There is no *single* input for f_2 that "explains" the effect of this behavior for all possible inputs of the row-player. Concretely, there is no input of f_2 that guarantees an output in $\{2, 3\}$.

The requirements for embedding are formalized in the following definition:

Definition 4. *For two functions α and β we say that α **leaks no more than** β if $\beta(y) = \beta(y') \Rightarrow \alpha(y) = \alpha(y')$ for all inputs y, y'. We say that α **refines** β if $\beta(y) \in \{\alpha(y), \perp\}$ for all inputs y.*

*Let $f : X \times Y \to Z$ and $g : \widehat{X} \times \widehat{Y} \to \widehat{Z}$. Without loss of generality, assume $f(x, y) = \mathsf{rect}_f(x, y)$ and $g(x, y) = \mathsf{rect}_g(x, y)$. We say that f **embeds in** g if:*

1. *(f appears as a submatrix in g) There exist two injective mappings, $A : X \to \widehat{X}$ and $B : Y \to \widehat{Y}$, and a third mapping, $C : \widehat{Z} \to Z \cup \{\perp\}$, such that $\forall x \in X, y \in Y : f(x, y) = C(g(A(x), B(y)))$.*

2. *(security guarantees) There exist mappings* $\widehat{A} : \widehat{X} \to X$ *and* $\widehat{B} : \widehat{Y} \to Y$ *such that the following hold:*
   (a) *(*g *doesn't reveal too much information)*
      – *for all* $\widehat{x} \in \widehat{X}$, $g(\widehat{x}, B(\cdot))$ *leaks no more than* $f(\widehat{A}(\widehat{x}), \cdot)$
      – *for all* $\widehat{y} \in \widehat{Y}$, $g(A(\cdot), \widehat{y})$ *leaks no more than* $f(\cdot, \widehat{B}(\widehat{y}))$
   (b) *(there are no ambiguous* g*-inputs)*
      – *for all* $\widehat{x} \in \widehat{X}$, $f(\widehat{A}(\widehat{x}), \cdot)$ *refines* $C(g(\widehat{x}, B(\cdot)))$.
      – *for all* $\widehat{y} \in \widehat{Y}$, $f(\cdot, \widehat{B}(\widehat{y}))$ *refines* $C(g(A(\cdot), \widehat{y}))$.

To understand this definition, it helps to see how the mappings $A, B, C, \widehat{A}, \widehat{B}$ relate to a secure protocol demonstrating $f \sqsubseteq_1 g$:

Lemma 5. *If* f *embeds in* g*, then* $f \sqsubseteq_1 g$ *via a deterministic protocol. This proves (4)* $\Rightarrow$ *[(1)* $\wedge$ *(2)* $\wedge$ *(3)] of Theorem 1 (stated in Section 2).*

Proof. Let f embed in g, with associated mappings as in Definition 4. The protocol for f is as follows:

– Alice sends input $A(x)$ to g where x is her f-input.
– Bob sends input $B(y)$ to g where y is his f-input.
– The parties both output $C(z)$ where z is the output they receive from g (they output $\perp$ if g gives output $\perp$).

Correctness follows from the first condition of Definition 4. Due to the symmetry in the definitions/protocol, we show security only against a malicious Alice.

Suppose Alice sends input $\widehat{x}$ to g. In the real protocol, Alice's view will consist of $g(\widehat{x}, B(y))$ and Bob's output will be $C(g(\widehat{x}, B(y)))$. In the ideal world, the simulator will do the following:

– The simulator sends $x^* = \widehat{A}(\widehat{x})$ to the ideal f, and obtains output $f(x^*, y) = C(g(A(x^*), B(y)))$.
– The simulator does not know Bob's input y but can choose any y' such that $f(x^*, y') = f(x^*, y)$. The simulator can give $g(\widehat{x}, B(y'))$ to Alice as her simulated view. From part 2a of Definition 4, we have that this is identical to the real view $g(\widehat{x}, B(y))$.
– The simulator checks whether $C(g(\widehat{x}, B(y))) = \perp$, and if so sends $(\text{DELIVER}, 0)$ to f. In this case, Bob's real and ideal outputs will both be $\perp$. Otherwise, it sends $(\text{DELIVER}, 1)$ to f and Bob will receive output $f(x^*, y)$.

Bob's ideal output is $f(x^*, y) = C(g(A(x^*), B(y)))$. From condition 2b of Definition 4, this matches the real output $C(g(\widehat{x}, B(y)))$.

Lemma 6. *For non-unilateral* f*, if* $f \sqsubseteq_1 g$ *via a deterministic protocol with simulation error*[5] *less than 1, then* f *embeds in* g*. This proves (3)* $\Rightarrow$ *(4) of Theorem 1 (stated in Section 2).*

[5] The simulation error of a protocol π is the maximum (supremum) over all environments of $|p_{\mathsf{real}} - p_{\mathsf{ideal}}|$, where p_{real} is the probability that the environment outputs 1 in the real interaction and p_{ideal} is the probability that it outputs 1 in the ideal interaction.

Proof (Sketch). The full proof is in Appendix B. The fact that $f \sqsubseteq_1 g$ by some deterministic protocol immediately reveals part 1 of the embedding: there must be some set of mappings that Alice and Bob use to map f-inputs to g-inputs and g-outputs to f-outputs.

The main technical portion of this proof is that, if simulator mappings $\widehat{A}$ and $\widehat{B}$ do not follow the rules in parts 2a and 2b of Definition 4, then the simulation error is 1, which contradicts our assertion that the simulation error is less than 1. The intuition for these attacks is clear from the examples above.

5 Instantaneous Protocols

In this section we show how to transform any secure protocol in the g-hybrid model into one that has an "instantaneous" property (described further in Section 5.4). The results in this section apply to arbitrary protocols. Later in Section 6 we give further transformations that are restricted to deterministic or logarithmic-round protocols.

5.1 Frontier Basics

Recall that $\Pr_\pi[t|xy]$ refers to the probability that the protocol results in transcript with prefix t, when run honestly on inputs x and y. We write $\Pr_\pi[\mathcal{E}|txy]$ to denote the probability that event $\mathcal{E}$ happens, given that the parties run honestly with inputs x and y, and conditioned on t being a prefix of the transcript.

Let F be any set of partial protocol transcripts, with the property that if $t \in F$, and t is a prefix of t', then $t' \in F$. In other words, F describes an event in the protocol that happens and does not "unhappen." In this case we call F a **frontier**, using the terminology of [17].

It is sometimes helpful to associate the frontier F with its set of prefix-minimal elements, as these represents transcript where *some condition happened for the first time.* Let $\mathsf{first}(F)$ denote the prefix-minimal elements of F.

If F is a frontier, we use notation $\Pr_\pi[F|xy]$ to denote the probability that F is encountered when running the protocol honestly on inputs x and y. More formally:

$$\Pr_\pi[F|xy] \stackrel{\text{def}}{=} \sum_{t \in \mathsf{first}(F)} \Pr_\pi[t|xy]$$

Finally, if F and G are two frontiers, then "$F < G$" denotes the event "either F happens strictly before G, or F happens and G never happens." More formally,

$$\Pr_\pi[F < G|xy] \stackrel{\text{def}}{=} \sum_{t \in \mathsf{first}(F \setminus G)} \Pr_\pi[t|xy]$$

5.2 Our Frontiers

Our analysis relies on two types of frontiers that we introduce:

$F^x_{\mathsf{A\text{-}ext}}$**:** captures the first time that the *simulator has extracted* with reasonable probability, in an ideal-world interaction involving a corrupt Alice running *honestly* on input x.

$F^x_{\mathsf{A\text{-}out}}$**:** captures the first time that Alice's output becomes relatively fixed, in the following sense. If the parties continue with honest behavior from such a point in the protocol, and Alice has input x, then Alice has only one likely output, no matter what Bob's input is.

We define such a frontier for every input x. We also define analogous frontiers with respect to Bob.

We have already defined $\Pr_\pi[\cdot|xy]$ notation with respect to an honest execution of the protocol on inputs x, y. Since $F^x_{\mathsf{A\text{-}ext}}$ refers to probabilities in an *ideal-model* interaction, we introduce notation to differentiate between the probabilities in real and ideal interactions. We write $\Pr_{\mathsf{A\text{-}sim}}[\cdot|xy]$ to refer to probabilities induced by an ideal-model interaction among malicious Alice running the protocol honestly on input x, the simulator for corrupt Alice, and ideal honest Bob with input y. $\Pr_{\mathsf{B\text{-}sim}}$ is defined analogously.

Definition 7. *At some point in an ideal interaction between corrupt Alice and the simulator, the simulator will at some point "extract" by sending an input to the ideal f. Define:*

$$\sigma_A(t,x) \stackrel{\text{def}}{=} \Pr_{A\text{-}sim}[\textit{simulator has previously extracted}|txy]$$

That is, $\sigma_A(t,x)$ is the probability that the simulator has extracted, given that the transcript so far is t. We define σ_B analogously.

In somewhat more detail, consider formally defining a simulator in terms of a its next-message function. Given as input its view so far (messages exchanged with the adversary and functionality and internal state), it outputs either $(\textsc{Prot}, m)$ to indicate sending a simulated protocol message m to the adversary, or $(\textsc{Ext}, x)$ to indicate sending an input x to the ideal functionality. Until the simulator talks to the ideal functionality, the only interaction is between the adversary and the simulator. As such, the simulator may be stateless for this period of time without loss of generality (by the same reasoning as in Section 3.3). The simulator's view certainly indicates whether extraction has happened (i.e., whether the view contains a $\textsc{Ext}$ message). Since the moment of extraction ($\textsc{Ext}$ message) is the first place that the simulator's view and the adversary-simulator transcript diverge, σ_{A} can be defined as a function of the transcript only.

Let $\sigma^*_{\mathsf{A}}(t,x)$ denote the probability that the simulator has decided to extract "at this instant," i.e., in response to the most recent protocol message sent by the adversary. Formally, let $t = (t_1, \ldots, t_n)$ be the partial transcript, where Alice is corrupt and speaks first:

$$\sigma^*_{\mathsf{A}}(t,x) \stackrel{\text{def}}{=} \Pr_\pi[t|x] \left(\prod_{\substack{i \text{ even} \\ i<n}} \Pr[\mathcal{S}(t_1 \cdots t_i) = (\textsc{Prot}, t_{i+1})] \right) \Pr[\mathcal{S}(t) = (\textsc{Ext}, \cdot)]$$

where $\mathcal{S}$ is the simulator's next-message function. Then

$$\sigma_{\mathsf{A}}(t,x) = \sum_{i<n} \sigma^*_{\mathsf{A}}\Big((t_1,\ldots,t_i),x\Big)$$

Note that before the simulator extracts, its view is perfectly independent of y in the ideal interaction. Its decision to extract, and hence the probability $\sigma_{\mathsf{A}}(t,x)$, depends only on x and not on y.

With that in mind, note that we have defined σ_{A} to refer to the probability that extraction has happened *strictly in the past* (note $i < n$ in the summation above). Another way to interpret σ_{A} is "the probability that the transcript might be affected by the honest party's input y." Hence, as the transcript evolves, the value of σ_{A} cannot change as a result of a message sent by Alice. It can only change as a result of a message generated by the simulator, hence an output of g or a simulated Bob-message.

Finally, note that our terminology considers when the simulator *actually* extracts, and not when the simulator has *in principle* enough information to extract. Again, the important issue is whether the simulator has already contacted the ideal functionality, and therefore the transcript may be influenced by the honest party's input.

Definition 8. *Given a secure protocol π with simulation error ε, define the following for all inputs x, y:*

$$F^x_{\mathsf{A\text{-}ext}} = \{t \mid \sigma_A(t,x) > 4\sqrt{\varepsilon}\}$$
$$F^y_{\mathsf{B\text{-}ext}} = \{t \mid \sigma_B(t,y) > 4\sqrt{\varepsilon}\}$$
$$F^x_{\mathsf{A\text{-}out}} = \{t \mid \forall y,y' : f(x,y) \neq f(x,y') \Rightarrow \min\left\{\begin{array}{c}\Pr_\pi[out\ f(x,y)|txy],\\ \Pr_\pi[out\ f(x,y')|txy']\end{array}\right\} < 1-\sqrt{\varepsilon}\}$$
$$F^y_{\mathsf{B\text{-}out}} = \{t \mid \forall x,x' : f(x,y) \neq f(x',y) \Rightarrow \min\left\{\begin{array}{c}\Pr_\pi[out\ f(x,y)|txy],\\ \Pr_\pi[out\ f(x',y)|tx'y]\end{array}\right\} < 1-\sqrt{\varepsilon}\}$$

Here $\Pr_\pi[out\ z|txy]$ refers to the probability that honest parties output z when starting the protocol at partial transcript t and running honestly with inputs x and y.

To understand $F^x_{\mathsf{A\text{-}out}}$, observe that for $t \in F^x_{\mathsf{A\text{-}out}}$ there is at most one output that can be induced with probability at least $1-\sqrt{\varepsilon}$. It may be the case that no valid output can be induced with this probability, in which case only $\perp$ output is likely from starting point t.

Note that if ε is a negligible function of the security parameter, then $\sqrt{\varepsilon}$ is a larger function but also still negligible.

5.3 Properties of the Frontiers

We now show that, roughly speaking, all the frontiers that we have defined must occur simultaneously, with overwhelming probability. Proofs for the lemmas in this section are given in Appendix C. Note that all lemmas hold with the roles of Alice and Bob reversed.

Lemma 9. *For all x, y:* $\Pr_\pi[F^x_{A\text{-}ext} < F^y_{B\text{-}out} \mid xy] < 2\sqrt{\varepsilon}$.

Proof (sketch). A partial transcript $t \in F^x_{\mathsf{A\text{-}ext}} \setminus F^y_{\mathsf{B\text{-}out}}$ represents a situation where there is reasonable probability that the simulator would have extracted from Alice in the ideal-interaction ($t \in F^x_{\mathsf{A\text{-}ext}}$), but in the real-interaction Alice can still induce two different outputs for Bob, each with good probability ($t \notin F^y_{\mathsf{B\text{-}out}}$). Intuitively, the simulator has extracted prematurely. This event should be rare.

Next we show that $F^x_{\mathsf{A\text{-}out}}$ is a point at which the honest parties can predict their eventual output.

Definition 10. *Fix x and let $t \in F^x_{A\text{-}out}$. Then there is at most one value z such that $\exists y : \Pr_\pi[out\ z | xyt] > 1 - \sqrt{\varepsilon}$. Let $\mathsf{guess}_A(t, x)$ denote this value z, and note that the value could be $\perp$. We extend the notation $\mathsf{guess}_A(t, x) = \perp$ in the case that $t \notin F^x_{A\text{-}out}$.*

Lemma 11. *For $z \neq \perp$ define $G^z = \{t \mid \mathsf{guess}_A(t, x) = z\}$. Then for all x, y:* $\Pr_\pi[G^{f(x,y)} | xy] > 1 - \varepsilon/2$. *Intuitively, upon reaching $F^x_{A\text{-}out}$, Alice can predict her eventual output with error at most $\varepsilon/2$.*

Lemma 12. *For all x, y (x not unilateral for f),* $\Pr_\pi[F^x_{A\text{-}out} < F^x_{A\text{-}ext} \mid xy] < 16\varepsilon$.

Proof (Sketch). A partial transcript $t \in F^x_{\mathsf{A\text{-}out}} \setminus F^x_{\mathsf{A\text{-}ext}}$ represents a situation where Alice can predict what the output will be ($t \in F^x_{\mathsf{A\text{-}out}}$), but the simulator probably has not extracted yet ($t \notin F^x_{\mathsf{A\text{-}ext}}$). This event should be rare, since in the ideal interaction Alice can gain no information about the f-output before the simulator extracts (assuming x is not a unilateral input, so that the output indeed depends on Bob's input).

Lemma 13. *For all x, y (y not unilateral),* $\Pr_\pi[F^y_{B\text{-}out} < F^x_{A\text{-}out} | xy] < 18\sqrt{\varepsilon}$.

Proof (Sketch). This follows from the fact that if $F^y_{\mathsf{B\text{-}out}} < F^x_{\mathsf{A\text{-}out}}$ then either $F^y_{\mathsf{B\text{-}out}} < F^y_{\mathsf{B\text{-}ext}}$ or $F^y_{\mathsf{B\text{-}ext}} < F^x_{\mathsf{A\text{-}out}}$, both of which are negligibly likely from Lemmas 9 & 12.

Lemma 14. *For all x, y', y (x, y' not unilateral),* $\Pr_\pi[F^{y'}_{B\text{-}out} < F^y_{B\text{-}out} | xy] < 42\sqrt{\varepsilon}$.

Proof (Sketch). If $F^{y'}_{\mathsf{B\text{-}out}} < F^y_{\mathsf{B\text{-}out}}$ then either $F^{y'}_{\mathsf{B\text{-}out}} < F^x_{\mathsf{A\text{-}ext}}$ or $F^x_{\mathsf{A\text{-}ext}} < F^y_{\mathsf{B\text{-}out}}$.

We can argue that the first case $F^{y'}_{\mathsf{B\text{-}out}} < F^x_{\mathsf{A\text{-}ext}}$ would be negligibly likely, if the parties run honestly on inputs x, y'. Unfortunately here we are using input y for Bob. But consider the ideal interaction with corrupt Alice. We are interested in an event in which the simulator is not likely to have extracted from Alice ($t \notin F^x_{\mathsf{A\text{-}ext}}$). Conditioned on the simulator not extracting, the protocol transcript is independent of Bob's input. Hence whatever is unlikely with input y' for Bob is also unlikely with input y for Bob.

The second case $F^x_{\mathsf{A\text{-}ext}} < F^y_{\mathsf{B\text{-}out}}$ is negligibly likely by Lemma 9.

5.4 Securely Truncating a Protocol

Lemma 15. *Let π be a secure protocol for f in the g-hybrid model. Define π' to be the following:*

- *On input x for Alice and y for Bob, both parties run π honestly on their given inputs.*
- *When the protocol transcript t reaches $F^{\tilde{x}}_{\mathsf{A\text{-}out}}$ for any $\tilde{x}$, or reaches $F^{\tilde{y}}_{\mathsf{B\text{-}out}}$ for any $\tilde{y}$, the parties terminate the protocol.*
- *Alice outputs $\mathsf{guess}_A(t, x)$ and Bob outputs $\mathsf{guess}_B(t, y)$.*

Then the truncated protocol π' is also a secure protocol for f.

Proof. Let ε denote the simulation error of π. First, we argue that π' is correct. Alice's output is $\mathsf{guess}_\mathsf{A}(t, x)$, which differs from the correct answer $f(x, y)$ only in the following events:

- $t \notin F^x_{\mathsf{A\text{-}out}}$ because the protocol reached $F^{x'}_{\mathsf{A\text{-}out}}$ and terminated strictly before reaching $F^x_{\mathsf{A\text{-}out}}$ for $x' \neq x$. By Lemma 14, this can happen only with probability $O(\sqrt{\varepsilon})$.
- $t \notin F^x_{\mathsf{A\text{-}out}}$ because the protocol reached $F^y_{\mathsf{B\text{-}out}}$ and terminated strictly before reaching $F^x_{\mathsf{A\text{-}out}}$. By Lemma 13, this can happen only with probability $O(\sqrt{\varepsilon})$.
- $t \in F^x_{\mathsf{A\text{-}out}}$ but $\mathsf{guess}_\mathsf{A}(t, x) \neq f(x, y)$. By Lemma 11, this can only happen with probability $O(\varepsilon)$.

As for security, the only difference between π and π' is that π' truncates early based on some condition. But this condition is public and *independent of either party's private inputs.* Hence the simulation for π' works as follows. It simply runs the simulator for π but terminates the protocol when the transcript reaches the public termination condition.

Overall π' is a secure protocol with negligible simulation error $O(\sqrt{\varepsilon})$.

Observe that the new protocol π' has the "instantaneous" property discussed in Section 2. Importantly for our purposes in the next section, with overwhelming probability $1 - O(\sqrt{\varepsilon})$ the protocol terminates on a transcript that is both in $F^x_{\mathsf{A\text{-}out}}$ and $F^y_{\mathsf{B\text{-}out}}$. Such a transcript must end with a message produced by the simulator in both ideal interactions (i.e., when either party is corrupt). Hence the last protocol message must be an output of g, with overwhelming probability.

6 Collapsing Protocols to a Single Call to g

We complete our main theorem with the following lemmas.

Lemma 16. *If, for incomplete and non-unilateral f and g, $f \sqsubseteq g$ via a protocol with strict upper bound on number of rounds $r = O(\log \kappa)$, then f embeds in g. This proves (1) ⇒ (4) of Theorem 1 (stated in Section 2).*

Proof (Sketch). The full proof is in Appendix D. Without loss of generality (from Lemma 15) the last step in π (in particular, the action in final round r) is a call to g with overwhelming probability.

We consider two cases. Consider a call to g that happens in the last round, following some partial transcript t. Imagine a new protocol where the parties simply "fast-forward" directly to this g-call by behaving as if the transcript so far was t. The result is a protocol consisting of a single call to g. If any call to g yields a *secure* protocol for f in this way, then we are done (we in fact have a *1-round* protocol for f).

In the other case, there may be no call to g during the final round of π that yields a secure protocol for f in this way. Intuitively, every time the protocol runs for the full r rounds there would have been a successful attack on the final call to g! Hence it must be negligibly unlikely that π would ever run for r rounds. We show that, in this case, truncating π after $r-1$ rounds results in a secure protocol for f.

We can repeatedly apply this argument at most $r-1$ times until we are guaranteed to obtain a 1-round protocol demonstrating $f \sqsubseteq_1 g$. The parameters are such that after truncating $r-1$ rounds, the resulting protocol has simulation error $c^{r-1}\sqrt{\varepsilon}$ for some constant c. Such a protocol is secure as long as $r = O(\log \kappa)$, since $c^{O(\log \kappa)}\sqrt{\varepsilon} = \mathsf{poly}(\kappa)\sqrt{\varepsilon}$, which is negligible.

Corollary 17. *If $f \sqsubseteq g$ via a deterministic protocol (of any number of rounds) then f embeds in g. This proves (2) ⇒ (4) of Theorem 1 (stated in Section 2).*

Proof. Deterministic protocols have zero simulation error (without loss of generality). Therefore, the same reasoning as in the previous proof applies but without any error accumulating with each round.

7 Tightness of the Characterization, Limitations

In this section we discuss why our main characterization does not extend (without modification) to consider unilateral functions or superlogarithmic-round, randomized protocols.

In Appendix E we discuss the possibility of extending our protocol model to allow parallel calls to g.

7.1 Unilateral Functions

Failure of our characterization on unilateral functions. In Figure 2 we give f and g which are unilateral. Bob is the column-player and thus has 2 unilateral inputs labeled B and C.

First, we argue that $f \not\sqsubseteq_1 g$. Suppose for sake of contradiction that such a protocol exists. Consider the simulator for a corrupt Bob who runs the protocol semi-honestly, on f-input that is chosen uniformly at random. The only message that the simulator sees is Bob's input to g, after which the simulator must extract an output to send to f. The simulator gets only one bit of information about Bob's input (as there are only 2 possible g-inputs), while there are 3 possibilities for the extracted f-input. It follows that with constant probability the simulator must extract the wrong input, and this error will be evident in the output of f.

	A	B	C
0	0	2	3
1	1	2	3

f

	A'	B'
0	0	2
1	1	2

g

Fig. 2. Unilateral functions violating the main theorem.

A	A	B
C	D	D
C	E	E

f

A	1	2	**A**	1	2	**B**	1	2
3	**A**	4	3	**A**	4	4	**B**	3
6	5	**A**	6	5	**A**	5	6	**B**
C	1	2	**D**	1	2	**D**	1	2
3	**C**	4	3	**D**	4	4	**D**	3
6	5	**C**	6	5	**D**	5	6	**D**
C	1	4	**E**	1	4	4	1	**E**
6	**C**	2	6	**E**	2	**E**	6	2
3	5	**C**	3	5	**E**	5	**E**	3

g

Fig. 3. Functions violating the main theorem via a superlogarithmic-round protocol. Note that the bottom-right 3×3 submatrix is unlike the others.

However, there is a simple protocol for f using g: Alice sends her f input directly to g. If Bob has f-input A, he should choose g-input A'. In this case, the parties will see that the g-output is in $\{0, 1\}$ and they terminate with this as their f-output. Otherwise, if Bob has f-input B or C, he should choose g-input B'. In this case, the parties will see that the g output is 2, and then Alice will wait for Bob to send a plain message containing either "2" or "3." Alice takes this message to be her output.

It is simple to see that this protocol is secure against a malicious Alice. For a malicious Bob, the simulator does the following. If Bob chooses g-input A', then the simulator extracts Bob's ideal f-input as A and simulates the g-output to equal the ideal f-output. If Bob chooses g-input B', then the simulator gives 2 as the simulated g-output, then waits for a message from Bob (either "2" or "3") and uses this as the extracted ideal f-input. The reason the simulation is secure is that in the second case (Bob chooses g-input B'), the fact that this is a unilateral input means that the simulator doesn't need to know Alice's input to perfectly simulate the g-output. Hence the simulator can delay extraction until the second protocol message, where intuitively Bob resolves which unilateral input he has.

Hence, we have $f \sqsubseteq g$ via a protocol consisting of a single call to g, plus (in some cases) one extra message. It is a deterministic, constant-round protocol, and yet $f \not\sqsubseteq_1 g$. This example shows that our classification does not extend to unilateral functions.

7.2 Deterministic / Logarithmic-Round Protocols

Consider the functions f and g in Figure 3. We first claim that f does not embed in g. Any embedding would map 3 f-columns into 3 distinct g-columns.[6] For any

[6] Perhaps columns are mapped to rows if the roles of Alice and Bob are swapped during the embedding. The analysis is the same for this scenario.

3 columns of g, there exists a row for which these columns have distinct entries – this is simple (albeit time-consuming) to verify. However, there is no row in f that has three distinct values. Hence the embedding would contradict rule 2a of the embedding definition. Concretely, any candidate protocol for $f \sqsubseteq_1 g$ would allow a corrupt row-player to learn the column-player's input in its entirety, which is not allowed by f.

However, there is a protocol for f that uses g. We group the rows and columns of g into groups of three, as distinguished by the dotted lines in the figure. Associate the first row of f with the first row group of g, etc. Similarly, associate the first column of f with the first column group of g, etc. The protocol for f is as follows:

- Alice chooses a g-input from the row group associated with her f-input, uniformly at random.
- Bob chooses a g-input from the column group associated with his f-input, uniformly at random.
- They call g with their selected g-inputs.
- If the output of the g-call was in {A, B, C, D, E}, terminate the protocol with that output. Otherwise, repeat (with fresh random choices for the g-inputs).

The correctness of this protocol is clear. By only sending g-inputs in the group associated with their f-inputs, each party restricts any terminating output of g to be one that was possible given their f-input.

To see that the protocol is secure, consider the following simulation. Suppose corrupt Alice chooses some g-input (row). With probability 1/3, the simulator decides that the protocol will terminate at this round. It converts the g-input to an f-input (according to its row group), sends that f-input to the ideal f, then simulates the g-output as the ideal f-input. With probability 2/3, the simulator decides that the protocol will continue. Note that in any row, there are 2 non-terminal g-outputs (for example, in the second row only 3 and 4 are possible), which are equally likely no matter which column group Bob has selected. The simulator simply chooses one of these two with equal probability as the simulated g-output. Then the same process repeats.

The parties' inputs will "match" by giving a terminal output with probability 1/3, meaning that the expected number of rounds is 3. The probability that the protocol continues for at least r rounds is $(2/3)^r$. We can get a protocol with a strict upper bound on round complexity by having the parties simply abort after some limit r number of rounds. If we set this limit as $r(\kappa) = \omega(\log \kappa)$, then the correctness of the protocol suffers by an amount $(2/3)^{\omega(\log \kappa)} = \kappa^{-\omega(1)}$, which is negligible. However, the simulation is still perfect, and the protocol is secure.

In summary, $f \sqsubseteq g$ via a randomized, (worst-case) superlogarithmic-round protocol, but f does not embed in g and so $f \not\sqsubseteq g$ via any deterministic protocol or any strict logarithmic-round protocol.

Acknowledgments

We thank the anonymous reviewers for their helpful comments.

References

1. D. Beaver. Perfect privacy for two-party protocols. In J. Feigenbaum and M. Merritt, editors, *Proceedings of DIMACS Workshop on Distributed Computing and Cryptography*, volume 2, pages 65–77. American Mathematical Society, 1989.
2. A. Beimel and T. Malkin. A quantitative approach to reductions in secure computation. In M. Naor, editor, *TCC 2004*, volume 2951 of *LNCS*, pages 238–257. Springer, Heidelberg, Feb. 2004.
3. M. Bellare and P. Rogaway. The security of triple encryption and a framework for code-based game-playing proofs. In S. Vaudenay, editor, *EUROCRYPT 2006*, volume 4004 of *LNCS*, pages 409–426. Springer, Heidelberg, May / June 2006.
4. R. Canetti. Universally composable security: A new paradigm for cryptographic protocols. In *42nd FOCS*, pages 136–145. IEEE Computer Society Press, Oct. 2001.
5. Y. Dodis and S. Micali. Lower bounds for oblivious transfer reductions. In J. Stern, editor, *EUROCRYPT'99*, volume 1592 of *LNCS*, pages 42–55. Springer, Heidelberg, May 1999.
6. S. D. Gordon, C. Hazay, J. Katz, and Y. Lindell. Complete fairness in secure two-party computation. In R. E. Ladner and C. Dwork, editors, *40th ACM STOC*, pages 413–422. ACM Press, May 2008.
7. Y. Ishai, M. Prabhakaran, and A. Sahai. Founding cryptography on oblivious transfer - efficiently. In D. Wagner, editor, *CRYPTO 2008*, volume 5157 of *LNCS*, pages 572–591. Springer, Heidelberg, Aug. 2008.
8. J. Kilian. Founding cryptography on oblivious transfer. In *20th ACM STOC*, pages 20–31. ACM Press, May 1988.
9. J. Kilian. *Uses of Randomness in Algorithms and Protocols*. PhD thesis, Department of Electrical Engineering and Computer Science, Mas sachusetts Institute of Technology, 1989.
10. J. Kilian. A general completeness theorem for two-party games. In *23rd ACM STOC*, pages 553–560. ACM Press, May 1991.
11. J. Kilian. More general completeness theorems for secure two-party computation. In *32nd ACM STOC*, pages 316–324. ACM Press, May 2000.
12. D. Kraschewski, H. K. Maji, M. Prabhakaran, and A. Sahai. A full characterization of completeness for two-party randomized function evaluation. In P. Q. Nguyen and E. Oswald, editors, *EUROCRYPT 2014*, volume 8441 of *LNCS*, pages 659–676. Springer, Heidelberg, May 2014.
13. D. Kraschewski and J. Müller-Quade. Completeness theorems with constructive proofs for finite deterministic 2-party functions. In Y. Ishai, editor, *TCC 2011*, volume 6597 of *LNCS*, pages 364–381. Springer, Heidelberg, Mar. 2011.
14. E. Kushilevitz. Privacy and communication complexity. In *30th FOCS*, pages 416–421. IEEE Computer Society Press, Oct. / Nov. 1989.
15. Y. Lindell and T. Rabin. Secure two-party computation with fairness - A necessary design principle. In Y. Kalai and L. Reyzin, editors, *TCC 2017, Part I*, volume 10677 of *LNCS*, pages 565–580. Springer, Heidelberg, Nov. 2017.
16. H. K. Maji, P. Ouppaphan, M. Prabhakaran, and M. Rosulek. Exploring the limits of common coins using frontier analysis of protocols. In Y. Ishai, editor, *TCC 2011*, volume 6597 of *LNCS*, pages 486–503. Springer, Heidelberg, Mar. 2011.
17. H. K. Maji, M. Prabhakaran, and M. Rosulek. Complexity of multi-party computation problems: The case of 2-party symmetric secure function evaluation. In O. Reingold, editor, *TCC 2009*, volume 5444 of *LNCS*, pages 256–273. Springer, Heidelberg, Mar. 2009.

18. H. K. Maji, M. Prabhakaran, and M. Rosulek. Cryptographic complexity classes and computational intractability assumptions. In A. C.-C. Yao, editor, *ICS 2010*, pages 266–289. Tsinghua University Press, Jan. 2010.
19. H. K. Maji, M. Prabhakaran, and M. Rosulek. A zero-one law for cryptographic complexity with respect to computational UC security. In T. Rabin, editor, *CRYPTO 2010*, volume 6223 of *LNCS*, pages 595–612. Springer, Heidelberg, Aug. 2010.
20. H. K. Maji, M. Prabhakaran, and M. Rosulek. A unified characterization of completeness and triviality for secure function evaluation. In S. D. Galbraith and M. Nandi, editors, *INDOCRYPT 2012*, volume 7668 of *LNCS*, pages 40–59. Springer, Heidelberg, Dec. 2012.
21. M. Prabhakaran and M. Rosulek. Cryptographic complexity of multi-party computation problems: Classifications and separations. In D. Wagner, editor, *CRYPTO 2008*, volume 5157 of *LNCS*, pages 262–279. Springer, Heidelberg, Aug. 2008.

A UC Security Recap

A 2-party SFE task is a deterministic function $f : X \times Y \to Z$. We identify f with the following ideal functionality: The functionality waits for input $x \in X$ from Alice and input $y \in Y$ from Bob. If no party is corrupt, the functionality gives output $f(x, y)$ to both parties. If any party is corrupt, the functionality gives $f(x, y)$ to the adversary and waits for a command $(\text{DELIVER}, b)$ from the adversary, where $b \in \{0, 1\}$. If $b = 0$, the functionality gives output $\perp$ to the honest party; if $b = 1$, the functionality gives $f(x, y)$ to the honest party. Hence, we consider **security with abort** — the functionality does not guarantee fairness or output delivery.

We assume basic familiarity with the UC framework, but briefly review the main concepts (specialized here for 2-party SFE). An execution in the framework begins with an *environment* $\mathcal{Z}$ (an arbitrary interactive TM) that chooses inputs for both of the *parties*. The parties interact with each other, and an ideal functionality for some function g, according to the protocol. The parties eventually give an output to the environment, who outputs a single bit. Throughout the entire interaction, there is an *adversary* $\mathcal{A}$ (an arbitrary interactive TM) who interacts arbitrarily with the environment. The adversary may also choose to *corrupt* one of the parties, which causes the party to come under complete control of the adversary. In that case, the party may deviate from the protocol. In this work we consider only **static** corruption, where the adversary must choose to corrupt a party before the protocol begins.

Given the description above, $\text{EXEC}[\pi, g, \mathcal{A}, \mathcal{Z}, 1^\kappa]$ denotes the probability that the environment outputs 1, when κ is the security parameter of the protocol.

A particular protocol of interest is the dummy protocol π_{dummy}. In this protocol, each party receives an input from the environment and sends it directly to the ideal functionality. When the ideal functionality delivers an output, the party gives it directly to the environment as output.

Definition 18. *We say f reduces to g, and write $f \sqsubseteq g$, if there exists a protocol π such that for all $\mathcal{A}$ there exists a $\mathcal{S}$ such that for all environments $\mathcal{Z}$, we have:*

$$\left|\text{EXEC}[\pi, g, \mathcal{A}, \mathcal{Z}, 1^\kappa] - \text{EXEC}[\pi_{\mathit{dummy}}, f, \mathcal{S}, \mathcal{Z}, 1^\kappa]\right| \text{ is negligible in } \kappa$$

We write $f \sqsubseteq_1 g$ if furthermore π has the following property: the parties make only one call to g and exchange no other messages.

In the execution of π_{dummy} we can think of $\mathcal{S}$ as simulating protocol π with $\mathcal{A}$. The security arises from the fact that the simulator communicates with the honest party via a single call to ideal f. This means that everything prior to the call $\mathcal{S}$ makes to ideal f must be independent of the honest party's input, and everything after that call must not affect the honest party's output. We call this event the *extraction*, as $\mathcal{S}$ must "extract" enough information about the input of $\mathcal{A}$ in order to correctly call f.

B Proofs for the Reducibility Characterization

Lemma (Restatement of Lemma 6). *For non-unilateral f, if $f \sqsubseteq_1 g$ via a deterministic protocol with simulation error less than 1, then f embeds in g.*

Proof. During the deterministic protocol, Alice and Bob both map their f-inputs to g-inputs and then immediately terminate the protocol, meaning that they were each able to map the g-output to the same f-output. The input maps are A and B in the embedding. The output map is C in the embedding.

By symmetry, we only consider security against a malicious Alice.

Clearly, A is injective. If it is not, then choose some pair (x, x') where $A(x) = A(x')$. Choose y so that $f(x, y) \neq f(x', y)$ – since f is not unilateral, such an input exists. Consider two environments – they ask the parties to honestly run the protocol with inputs x, y and x', y respectively, and return 1 if the output is correct. In the real world, in both cases the parties will return the same output, so one of them is incorrect. Therefore, one of the environments has simulation error 1.

Because this protocol is UC-secure, there must be some method by which the simulator takes the parties' g-inputs and translate them to f-inputs upon extraction. Call these mappings for Alice and Bob $\widehat{A}$ and $\widehat{B}$, respectively. It suffices to show that these mappings satisfy 2a and 2b of Definition 4. We show that if the mappings violate Definition 4, then the simulation error of the protocol is 1.

(2a) Assume that $\widehat{A}$ violates part 2a. That is, there exists an $\widehat{x}$ where it is not the case that $g(\widehat{x}, B(\cdot))$ leaks no more than $f(\widehat{A}(\widehat{x}), \cdot)$. In particular, there is some pair (y, y') where $f(\widehat{A}(\widehat{x}), y) = f(\widehat{A}(\widehat{x}), y')$ but $g(\widehat{x}, B(y)) \neq g(\widehat{x}, B(y'))$.

Consider two environments:

– Adversary Alice uses input $\widehat{x}$, honest Bob uses input y, return 1 if Alice's view is $g(\widehat{x}, B(y))$.

– Adversary Alice uses input $\widehat{x}$, honest Bob uses input y', return 1 if Alice's view is $g(\widehat{x}, B(y'))$.

In the real world, both environments output 1 with probability 1. In the ideal world, the simulator's view in both environments is $f(\widehat{A}(\widehat{x}), y) = f(\widehat{A}(\widehat{x}), y')$. Since the simulator is deterministic, it must give the same simulated g-output for both environments. However, $g(\widehat{x}, B(y)) \neq g(\widehat{x}, B(y'))$, so in at least one of the environments the probability of outputting 1 is 0. Therefore, the simulation error is 1.

(2b) Assume that $\widehat{A}$ violates part 2b. Then there exists some $\widehat{x}$ and y where $C(g(\widehat{x}, B(y))) \notin \{f(\widehat{A}(\widehat{x}), y), \perp\}$. Consider the environment in which corrupt Alice uses g-input $\widehat{x}$ and honest Bob uses input y, and the environment outputs 1 if Bob's output is $C(g(\widehat{x}, B(y)))$. The environment outputs 1 with probability 1 in the real world. But in the ideal world, Bob's output is either $f(\widehat{A}(\widehat{x}), y)$ or $\perp$. This environment demonstrates a simulation error of 1.

C Proofs for Frontier Properties

Lemma (Restatement of Lemma 9). *For all x, y:* $\Pr_\pi[F^x_{\mathsf{A\text{-}ext}} < F^y_{\mathsf{B\text{-}out}} \mid xy] < 2\sqrt{\varepsilon}$.

Proof. Let $\mathsf{bad} = \mathsf{first}(F^x_{\mathsf{A\text{-}ext}} \setminus F^y_{\mathsf{B\text{-}out}})$, whose probability we wish to bound. A partial transcript $t \in \mathsf{bad}$ represents a situation where there is reasonable probability that a simulator would have extracted an effective input for Alice ($t \in F^x_{\mathsf{A\text{-}ext}}$), but in the protocol Alice can still induce two different outputs for Bob, each with good probability ($t \notin F^y_{\mathsf{B\text{-}out}}$). Intuitively, the simulator has extracted prematurely. This event should be rare.

Consider the following strategy for corrupt Alice and environment:

– Run the protocol with input y for honest Bob, and Alice initially behaving semi-honestly with input x.
– If the protocol transcript avoids bad, then the adversary gives up and the environment outputs 0.
– Otherwise, when the partial transcript reaches $t \in \mathsf{bad}$ for the first time, then the properties of bad guarantee that there are two values x_0, x_1 such that $f(x_0, y) \neq f(x_1, y)$ and $\Pr_\pi[\text{out } f(x_c, y) | t x_c y] \geq 1 - \sqrt{\varepsilon}$ for both $c \in \{0, 1\}$.
– The adversary sends x_0, x_1 to the environment, who chooses a random $c \leftarrow \{0, 1\}$.
– The adversary switches strategies to run the protocol honestly with input x_c. The environment outputs 1 if Bob's eventual output is $f(x_c, y)$. Otherwise the environment outputs 0.

Let succ denote the event that the environment outputs 1.

In the real interaction, the environment outputs 1 only when the transcript hits bad and the adversary is successful in forcing Bob's output, which happens with probability at least $1 - \sqrt{\varepsilon}$ by the properties of bad. So:

$$\Pr_\pi[\mathsf{succ}] \geq \Pr_\pi[\mathsf{bad}|xy](1 - \sqrt{\varepsilon})$$

In the ideal interaction (between the adversary and simulator), Bob's output is now determined differently, as the output of the ideal f. There are two ways the environment outputs 0: (1) when the simulated transcript avoids bad; (2) when the transcript reaches bad but the simulator has already extracted. In the latter case, the environment's choice of c is independent of the simulator's extraction, so with further probability at least $1/2$ the honest Bob will not output $f(x_c, y)$. So:

$$\Pr_{\mathsf{A\text{-}sim}}[\mathsf{succ}] \leq (1 - (4\sqrt{\varepsilon})/2) \Pr_{\mathsf{A\text{-}sim}}[\mathsf{bad}|xy]$$

From the security of the protocol:

$$|\Pr_\pi[\mathsf{succ}] - \Pr_{\mathsf{A\text{-}sim}}[\mathsf{succ}]| < \varepsilon$$
$$|\Pr_\pi[\mathsf{bad}|xy] - \Pr_{\mathsf{A\text{-}sim}}[\mathsf{bad}|xy]| < \varepsilon$$

Hence:

$$\begin{aligned}
\varepsilon &> \Pr_\pi[\mathsf{succ}] - \Pr_{\mathsf{A\text{-}sim}}[\mathsf{succ}] \\
&\geq \Pr_\pi[\mathsf{bad}|xy](1 - \sqrt{\varepsilon}) - (1 - 2\sqrt{\varepsilon}) \Pr_{\mathsf{A\text{-}sim}}[\mathsf{bad}|xy] \\
&\geq \Pr_\pi[\mathsf{bad}|xy](1 - \sqrt{\varepsilon}) - (1 - 2\sqrt{\varepsilon})(\Pr_\pi[\mathsf{bad}|xy] + \varepsilon) \\
&= \Pr_\pi[\mathsf{bad}|xy]\sqrt{\varepsilon} - \varepsilon(1 - 2\sqrt{\varepsilon}) \\
&> \Pr_\pi[\mathsf{bad}|xy]\sqrt{\varepsilon} - \varepsilon
\end{aligned}$$

Solving for $\Pr_\pi[\mathsf{bad}|xy]$:

$$\Pr_\pi[\mathsf{bad}|xy] < \frac{2\varepsilon}{\sqrt{\varepsilon}} = 2\sqrt{\varepsilon}$$

Lemma (Restatement of Lemma 11). *For $z \neq \perp$ define $G^z = \{t \mid \mathsf{guess}_A(t, x) = z\}$. Then for all x, y: $\Pr_\pi[G^{f(x,y)}|xy] > 1 - \varepsilon/2$. Intuitively, upon reaching $F^x_{\mathsf{A\text{-}out}}$, Alice can predict her eventual output with error at most $\varepsilon/2$.*

Proof. Define $\mathsf{bad} = F^x_{\mathsf{A\text{-}out}} \setminus G^{f(x,y)}$. Intuitively, these are the places in the protocol where $\mathsf{guess}_\mathsf{A}(t, x) \neq f(x, y)$.

From the correctness of the protocol, we have:

$$\begin{aligned}
\varepsilon &> \Pr_\pi[\text{output not } f(x,y)|xy] \\
&\geq \sum_{t \in \mathsf{first}(\mathsf{bad})} \Pr_\pi[t|xy] \Pr_\pi[\text{out } \mathsf{guess}_\mathsf{A}(t, x)|txy] \\
&\geq \sum_{t \in \mathsf{first}(\mathsf{bad})} \Pr_\pi[t|xy](1 - \sqrt{\varepsilon}) = (1 - \Pr_\pi[G^{f(x,y)}|xy])(1 - \sqrt{\varepsilon})
\end{aligned}$$

Solving for the probability expression:

$$\Pr_\pi[G^{f(x,y)}|xy] \geq 1 - \tfrac{\varepsilon}{1-\sqrt{\varepsilon}} > 1 - \varepsilon/2$$

Lemma (Restatement of Lemma 12). *For all x, y, if x is not a unilateral input for f, then $\Pr_\pi[F^x_{\mathsf{A\text{-}out}} < F^x_{\mathsf{A\text{-}ext}} \mid xy] < 16\varepsilon$.*

Proof. Let $\mathsf{bad} = \mathsf{first}(F^x_{\mathsf{A\text{-}out}} \setminus F^x_{\mathsf{A\text{-}ext}})$, whose probability we wish to bound. A partial transcript $t \in \mathsf{bad}$ represents a situation where Alice can predict what the output will be ($t \in F^x_{\mathsf{A\text{-}out}}$), but the simulator probably has not extracted yet ($t \notin F^x_{\mathsf{A\text{-}ext}}$). This event should be rare, since in the ideal world Alice can gain no information about the f-output before the simulator extracts.

Let x, y be given as in the premise of the lemma. Since x is not a unilateral input, let y' be such that $f(x, y) \neq f(x, y')$. Consider the following interaction with a corrupt Alice and environment:

- Alice initially runs the protocol honestly with input x. The environment randomly chooses input $y^* \leftarrow \{y, y'\}$ for Bob. If the transcript avoids bad then the adversary gives up and the environment outputs 0.
- Otherwise, if the partial transcript reaches $t \in \mathsf{bad}$, there is a unique $z = \mathsf{guess}_\mathsf{A}(t, x)$ such that $\Pr_\pi[\text{out } z | tx] > 1 - \sqrt{\varepsilon}$. The adversary reports z to the environment.
- The environment outputs 1 if $z = f(x, y^*)$.

Let succ denote the probability that the environment outputs 1.

In the real interaction, the environment outputs 0 only if the transcript avoids bad or if $\mathsf{guess}_\mathsf{A}(t, x)$ is incorrect. By the union bound and Lemma 11,

$$\Pr_\pi[\neg\mathsf{succ}] \leq \Pr_\pi[\neg\mathsf{bad} | xy^*] + \Pr_\pi[\neg G^{f(x,y^*)} | xy^*] \leq 1 - \Pr_\pi[\mathsf{bad} | xy^*] + \varepsilon/2$$

In the ideal interaction, the environment outputs 0 in the following (mutually exclusive) scenarios: (1) the simulated transcript avoids bad; (2) the transcript reaches bad and the simulator has not yet extracted. In the latter case, the adversary's view is independent of the environment's choice of y^*, and so the environment outputs 0 with probability at least $1/2$. Hence:

$$\begin{aligned}
\Pr_{\mathsf{A\text{-}sim}}[\neg\mathsf{succ}] &\geq \Pr_{\mathsf{A\text{-}sim}}[\neg\mathsf{bad} | xy^*] + \Pr_{\mathsf{A\text{-}sim}}[\mathsf{bad} \wedge \text{no extract} | xy^*]/2 \\
&\geq 1 - \Pr_{\mathsf{A\text{-}sim}}[\mathsf{bad} | xy^*] + \Pr_{\mathsf{A\text{-}sim}}[\mathsf{bad} | xy^*](1 - 4\sqrt{\varepsilon})/2 \\
&= 1 - \Pr_{\mathsf{A\text{-}sim}}[\mathsf{bad} | xy^*](\tfrac{1}{2} + 2\sqrt{\varepsilon})
\end{aligned}$$

Combining:

$$\begin{aligned}
\varepsilon &> \Pr_\pi[\mathsf{succ}] - \Pr_{\mathsf{A\text{-}sim}}[\mathsf{succ}] \\
&\geq \Pr_\pi[\mathsf{bad} | xy^*] - \varepsilon/2 - \Pr_{\mathsf{A\text{-}sim}}[\mathsf{bad} | xy^*](\tfrac{1}{2} + 2\sqrt{\varepsilon}) \\
&\geq \Pr_\pi[\mathsf{bad} | xy^*] - \varepsilon/2 - (\Pr_\pi[\mathsf{bad} | xy^*] + \varepsilon)(\tfrac{1}{2} + 2\sqrt{\varepsilon}) \\
&= \Pr_\pi[\mathsf{bad} | xy^*](\tfrac{1}{2} - 2\sqrt{\varepsilon}) - \varepsilon(1 - 2\sqrt{\varepsilon})
\end{aligned}$$

Solving for the probability expression:

$$\Pr_\pi[\mathsf{bad} | xy^*] \leq \frac{\varepsilon(2 - 2\sqrt{\varepsilon})}{(\frac{1}{2} - 2\sqrt{\varepsilon})} < \frac{2\varepsilon}{1/4} = 8\varepsilon$$

Since $\Pr_\pi[\mathsf{bad} | xy^*]$ is the average of $\Pr_\pi[\mathsf{bad} | xy]$ and $\Pr_\pi[\mathsf{bad} | xy']$, it follows that $\Pr_\pi[\mathsf{bad} | xy] < 16\varepsilon$.

Lemma (Restatement of Lemma 13). *For all x, y, if y is not a unilateral input, then $\Pr_\pi[F^y_{B\text{-}out} < F^x_{A\text{-}out}|xy] < 18\sqrt{\varepsilon}$.*

Proof. By a union bound,

$$\begin{aligned}\Pr_\pi[F^y_{\mathsf{B\text{-}out}} < F^x_{\mathsf{A\text{-}out}}|xy] &\le \Pr_\pi[F^y_{\mathsf{B\text{-}out}} < F^y_{\mathsf{B\text{-}ext}}|xy] + \Pr_\pi[F^y_{\mathsf{B\text{-}ext}} < F^x_{\mathsf{A\text{-}out}}|xy] \\ &\le 16\varepsilon + 2\sqrt{\varepsilon} < 18\sqrt{\varepsilon}\end{aligned}$$

Before proving Lemma 14, we introduce two helper lemmas:

Lemma 19. *For all x, y, if neither x nor y are a unilateral input, then $\Pr_\pi[F^y_{B\text{-}out} < F^x_{A\text{-}ext}|xy] < 34\sqrt{\varepsilon}$.*

Proof. By a union bound,

$$\begin{aligned}\Pr_\pi[F^y_{\mathsf{B\text{-}out}} < F^x_{\mathsf{A\text{-}ext}}|xy] &\le \Pr_\pi[F^y_{\mathsf{B\text{-}out}} < F^y_{\mathsf{B\text{-}ext}}|xy] \\ &\quad + \Pr_\pi[F^y_{\mathsf{B\text{-}ext}} < F^x_{\mathsf{A\text{-}out}}|xy] \\ &\quad + \Pr_\pi[F^x_{\mathsf{A\text{-}out}} < F^x_{\mathsf{A\text{-}ext}}|xy] \\ &\le 16\varepsilon + 2\sqrt{\varepsilon} + 16\varepsilon < 34\sqrt{\varepsilon}\end{aligned}$$

Lemma 20. *Let F be any frontier in the protocol. For all x, y, y',*

$$\left|\Pr_\pi[F < F^x_{A\text{-}ext}|xy] - \Pr_\pi[F < F^x_{A\text{-}ext}|xy']\right| < 6\sqrt{\varepsilon}$$

Proof. Let $G = F \setminus F^x_{\mathsf{A\text{-}ext}}$. The main idea is that in the ideal interaction with corrupt Alice, it is unlikely that the simulator has extracted before the protocol has reached G. Conditioned on the simulator not yet extracting, the transcript is completely independent of Bob's input.

Consider running the ideal interaction and halting it when the transcript reaches either F or $F^x_{\mathsf{A\text{-}ext}}$. Halting at this point is sufficient to determine whether the event $F < F^x_{\mathsf{A\text{-}ext}}$ happened. We obtain two interactions depending on whether Bob is given input y or y'. In the terminology of Bellare-Rogaway [3], these are two *identical-until-bad games*, where the "bad" event is that the simulator extracts but $F^x_{\mathsf{A\text{-}ext}}$ is not immediately reached. By the definition of $F^x_{\mathsf{A\text{-}ext}}$, the bad event happens with probability at most $4\sqrt{\varepsilon}$. By the lemma in [3], this probability of the bad event bounds the distinguishing bias between the two games.

Then applying the security of the protocol we have:

$$\begin{aligned}&\left|\Pr_\pi[F < F^x_{\mathsf{A\text{-}ext}}|xy] - \Pr_\pi[F < F^x_{\mathsf{A\text{-}ext}}|xy']\right| \\ &\qquad \le \left|\Pr_{\mathsf{A\text{-}sim}}[F < F^x_{\mathsf{A\text{-}ext}}|xy] - \Pr_{\mathsf{A\text{-}sim}}[F < F^x_{\mathsf{A\text{-}ext}}|xy']\right| + 2\varepsilon \\ &\qquad < 4\sqrt{\varepsilon} + 2\varepsilon < 6\sqrt{\varepsilon}\end{aligned}$$

Lemma (Restatement of Lemma 14). *For all x, y', y, with x, y' not unilateral, $\Pr_\pi[F^{y'}_{B\text{-}out} < F^y_{B\text{-}out}|xy] < 42\sqrt{\varepsilon}$.*

Proof. Let $\mathsf{bad} = \mathsf{first}(F^{y'}_{\mathsf{B\text{-}out}} \setminus F^{y}_{\mathsf{B\text{-}out}})$, whose probability we wish to bound. We partition bad into two parts: $\mathsf{bad}_1 = \mathsf{bad} \cap F^{x}_{\mathsf{A\text{-}ext}}$ and $\mathsf{bad}_2 = \mathsf{bad} \setminus F^{x}_{\mathsf{A\text{-}ext}}$.

Since $\mathsf{bad}_1 \subseteq F^{x}_{\mathsf{A\text{-}ext}} \setminus F^{y}_{\mathsf{B\text{-}out}}$ (i.e., the event $F^{x}_{\mathsf{A\text{-}ext}} < F^{y}_{\mathsf{B\text{-}out}}$ is true for these transcripts), Lemma 9 implies that

$$\Pr_\pi[\mathsf{bad}_1|xy] < 2\sqrt{\varepsilon}.$$

Since bad_2 happens strictly before the $F^{x}_{\mathsf{A\text{-}ext}}$ event, Lemma 20 implies that

$$\left|\Pr_\pi[\mathsf{bad}_2|xy] - \Pr_\pi[\mathsf{bad}_2|xy']\right| < 6\sqrt{\varepsilon}.$$

Since $\mathsf{bad}_2 \subseteq F^{y'}_{\mathsf{B\text{-}out}} \setminus F^{x}_{\mathsf{A\text{-}ext}}$, Lemma 19 implies that

$$\Pr_\pi[\mathsf{bad}_2|xy'] < 34\sqrt{\varepsilon}.$$

Putting everything together, we have:

$$\begin{aligned}\Pr_\pi[\mathsf{bad}|xy] &\leq \Pr_\pi[\mathsf{bad}_1|xy] + \Pr_\pi[\mathsf{bad}_2|xy] \\ &< 2\sqrt{\varepsilon} + \Pr_\pi[\mathsf{bad}_2|xy'] + 6\sqrt{\varepsilon} \\ &< 2\sqrt{\varepsilon} + 34\sqrt{\varepsilon} + 6\sqrt{\varepsilon} \\ &= 42\sqrt{\varepsilon}\end{aligned}$$

D Proofs for Collapsing Protocols

In order to collapse a protocol to a single round, we use two important properties of instantaneous protocols. First, by Lemma 15 we can consider only protocols that end with a call to g. Second, by Lemma 20 before the final call to g the parties' inputs do not have a noticeable effect on the distribution of transcripts.

Lemma 21. *For all f and g there is a constant $c_{f,g}$ such that if $f \sqsubseteq_1 g$ via a protocol π with simulation error ε, then $f \sqsubseteq_1 g$ via a* deterministic *protocol π' with simulation error at most $c_{f,g}\varepsilon$.*

Proof. Since π consists of only one call to g, the only choices Alice, Bob, and the simulator can make in the protocol are:

- The mapping of Alice's f-input to her g-input
- The mapping of Bob's f-input to his g-input
- The mappings of either party's g-input to a suitable f-input in the simulator
- The mapping of the g-output to an f-output

The only ways that randomness can manifest in the protocol are in the choice of these mappings.

Let $c_{f,g}$ be the number of possible combinations of such mappings based on these random coins. This is certainly a constant, although it is perhaps very large.

Select the mapping combination that was most likely to be chosen in π. Consider the deterministic protocol π' constructed by locking in these choices at the start of the protocol. The probability that Alice, Bob, and the simulator in π match the behavior of π' is at least $1/c_{f,g}$. Then, if the simulation error of π' is δ, the simulation error of π must be at least $\delta/c_{f,g}$. Therefore, if π has simulation error ε, then π' must have simulation error at most $c_{f,g}\varepsilon$.

Given a protocol π with a strict upper limit of r rounds, $\mathsf{trunc}(\pi, i)$ is the protocol constructed by truncating π after $r - i$ rounds, outputting $\perp$ if π was not finished. Note that $\mathsf{trunc}(\pi, 0) = \pi$.

Let $\mathcal{R}$ be the transcripts of $\mathsf{trunc}(\pi, i)$ which are $r - i - 1$ rounds (that is, there is one action to go in the protocol) but have not terminated yet.

Lemma 22. *If π has simulation error ε, then for all x, y, y' and all i:*

$$\left| \Pr_{\mathsf{trunc}(\pi,i)}[\mathcal{R}|xy] - \Pr_{\mathsf{trunc}(\pi,i)}[\mathcal{R}|xy'] \right| < 6\sqrt{\varepsilon}$$

Proof. By Lemma 20 (in Appendix C) we know that this is true in $\mathsf{trunc}(\pi, 0) = \pi$, as these $\mathcal{R}$ transcripts are strictly before $F^x_{\mathsf{A\text{-}ext}}$. Truncating doesn't change simulator extraction probabilities, as the simulator for $\mathsf{trunc}(\pi, i)$ just runs the simulator for π up until the truncated transcript. Therefore, the lemma still holds with respect to $\mathsf{trunc}(\pi, i)$.

Lemma 23. *If a protocol π is not ε-secure against malicious Alice, then there is an environment ENV for π with the following properties:*

1. ENV *chooses inputs for Bob uniformly at random*
2. $\Pr_\pi[\mathsf{ENV} \text{ outputs } 1] > \frac{1}{2} + \frac{\varepsilon}{4|Y|}$.
3. $\Pr_{\mathsf{A\text{-}sim}}[\mathsf{ENV} \text{ outputs } 1] < \frac{1}{2} - \frac{\varepsilon}{4|Y|}$

Of course, a symmetrical lemma holds for protocol π that is insecure against malicious Bob.

Proof. Take an environment ENV_0 for which π has simulation error ε. Construct ENV as follows.

- Choose Bob's input y^* uniformly at random.
- Let $p(y)$ be the probability that ENV_0 chooses y. Let p_{max} be the maximum $p(y)$.
- Flip a coin with probability $1-(p(y^*)/p_{\mathsf{max}})$. If it comes up heads, abort and return 0.
- Otherwise, run ENV_0.

Note that $\mathsf{ENV} \equiv \mathsf{ENV}_0$ conditioned on ENV not aborting. We abort with probability at most $(|Y|-1)/|Y|$ (as we never abort for the y where $p(y) = p_{\mathsf{max}}$). $(|Y|-1)/|Y|$ is a constant. The simulation error of ENV is therefore at least $\varepsilon/|Y|$.

At this point, possibly invert the output of ENV such that $\Pr_\pi[\mathsf{ENV} \text{ outputs } 1]$ is greater than $\Pr_{\mathsf{A\text{-}sim}}[\mathsf{ENV} \text{ outputs } 1]$. This will not affect the simulation error.

The probability of ENV returning 1 in the real or simulated environments differs by $\varepsilon/|Y|$, and is centered around some constant p. That is, the probabilities are at least $p+\delta$ and at most $p-\delta$ respectively, where $\delta = \varepsilon/(2|Y|)$. Perform the following operations to ensure that the probability is centered around $1/2$ as required by the lemma.

1. If $p > 1/2$, run normally with probability $\frac{1}{2p}$. Otherwise, return 0.
2. if $p < 1/2$, run normally with probability $\frac{1}{2(1-p)}$. Otherwise, return 1.

This will "normalize" the average of the probabilities in the real and ideal world to $1/2$. This might shrink δ slightly – up to a factor of 2. The minimum value of δ is $\frac{\varepsilon}{4|Y|}$, as required by the lemma.

Define $\varepsilon_0 = \sqrt{\varepsilon}$ and $\varepsilon_i = (52nc)\varepsilon_{i-1} = (52nc)^i\varepsilon_0$ where $c = c_{f,g}$ is the constant defined in Lemma 21 and n is the maximum of $|X|$ and $|Y|$.

Lemma 24. *If* $\mathsf{trunc}(\pi, i)$ *has simulation error at most* ε_i *then either* f *embeds in* g *or* $\mathsf{trunc}(\pi, i+1)$ *has simulation error at most* ε_{i+1}.

Proof. Consider any partial transcript t where the next action in $\mathsf{trunc}(\pi, i)$ is for the parties to make a call to g. Let protocol π_t be defined as follows: the parties "fast-forward" to t by imagining the transcript up to that round. They then complete the call to g and exit immediately afterwards.

If there is any π_t with simulation error less than $1/c$ (call such a t *good*), f embeds in g: by Lemma 21, there exists a single-round deterministic protocol for f in a g-hybrid world with simulation error less than 1. Then, by Lemma 6, f embeds in g.

Assume that there is no good t in round $r - i - 1$ of $\mathsf{trunc}(\pi, i)$ (that is, in $\mathcal{R}$). We wish to bound the probability $\Pr_{\mathsf{trunc}(\pi,i)}[\mathcal{R}|xy]$ for all x, y in this case. Note that some $t \in \mathcal{R}$ may have simulation error when run against one malicious party but not the other. Let $\mathcal{R}_A$ be those t which have unacceptable simulation error against Alice, and let $\mathcal{R}_B$ be similarly defined for Bob. Then:

$$\Pr_{\mathsf{trunc}(\pi,i)}[\mathcal{R}|xy] \leq \Pr_{\mathsf{trunc}(\pi,i)}[\mathcal{R}_A|xy] + \Pr_{\mathsf{trunc}(\pi,i)}[\mathcal{R}_B|xy]$$

Consider, then, the probability $\Pr_{\mathsf{trunc}(\pi,i)}[\mathcal{R}_A|xy]$. A symmetric argument will work for $\mathcal{R}_B$, and therefore we can use these to get a bound on the overall probability $\Pr_{\mathsf{trunc}(\pi,i)}[\mathcal{R}|xy]$.

For each t in $\mathcal{R}$, consider π_t. The simulation error is at least $1/c$, so by Lemma 23 we can construct an environment ENV_t satisfying:

1. ENV_t chooses inputs for Bob uniformly at random
2. $\Pr_{\pi_t}[\mathsf{ENV}_t \text{ outputs } 1] > \frac{1}{2} + \frac{1}{4|Y|c}$
3. $\Pr_{\mathsf{A\text{-}sim\text{-}for\text{-}}\pi_t}[\mathsf{ENV}_t \text{ outputs } 1] < \frac{1}{2} - \frac{1}{4|Y|c}$

Fix a particular x and y. Consider the following attack:

– Alice runs on x, and the environment chooses input y^* for Bob uniformly at random.
– If the parties reach $t \in \mathcal{R}$, let ENV_t sample an input for Bob. If it samples y^*, run ENV_t, possibly allowing Alice to change her input maliciously. Otherwise, output 0.

The probability that during this attack we output 1 is the probability that we reached $\mathcal{R}$ times the probability that ENV_t chooses input y^* times the probability that ENV_t succeeds at its attack given y^*. These probabilities are independent.

$$\sum_{t \in \mathcal{R}_A} \sum_{y^*} \Pr_{\mathsf{trunc}(\pi,i)}[t|xy^*] \frac{1}{|Y|} \Pr_{\pi_t}[\mathsf{ENV}_t \text{ returns } 1|y^*]$$

If we eliminate the summations, we get the following expression:

$$\left(\Pr_{\mathsf{trunc}(\pi,i)}[\mathcal{R}_A|xy^*]\right)\left(\frac{1}{2} + \frac{1}{4|Y|c}\right)$$

By Lemma 22 we can replace y^* with the input for Bob we desire to bound probability against, y, with a small change in probability of reaching a transcript.

The probability the environment outputs 1 in the real world is therefore at least:

$$\left(\Pr_{\mathsf{trunc}(\pi,i)}[\mathcal{R}_A|xy] - 6\sqrt{\varepsilon}\right)\left(\frac{1}{2} + \frac{1}{4|Y|c}\right)$$

By a similar series of arguments, the probability the environment outputs 1 in the ideal world is at most:

$$\left(\Pr_{\mathsf{A\text{-}sim\text{-}for\text{-}trunc}(\pi,i)}[\mathcal{R}_A|xy] + 6\sqrt{\varepsilon}\right)\left(\frac{1}{2} - \frac{1}{4|Y|c}\right)$$

Because the simulation error of $\mathsf{trunc}(\pi, i)$ is ε_i, the above expression is at most the following value:

$$\left(\Pr_{\mathsf{trunc}(\pi,i)}[\mathcal{R}_A|xy] + 6\sqrt{\varepsilon} + \varepsilon_i\right)\left(\frac{1}{2} - \frac{1}{4|Y|c}\right)$$

We know that, because $\mathsf{trunc}(\pi, i)$ has a simulation error of ε_i, the difference between the output probabilities in the real and ideal worlds is at most ε_i, which means that, in particular:

$$\begin{aligned}
\varepsilon_i &\geq \Pr_{\mathsf{trunc}(\pi,i)}[\mathcal{R}_A|xy]\frac{1}{2|Y|c} - 12\sqrt{\varepsilon} - \varepsilon_i\left(\frac{1}{2} - \frac{1}{4|Y|c}\right) \\
&\Rightarrow \Pr_{\mathsf{trunc}(\pi,i)}[\mathcal{R}_A|xy] \leq \left(|Y|c + \frac{1}{2}\right)\varepsilon_i + 24|Y|c\sqrt{\varepsilon} \\
&\Rightarrow \Pr_{\mathsf{trunc}(\pi,i)}[\mathcal{R}_A|xy] \leq (26|Y|c)\,\varepsilon_i
\end{aligned}$$

Recall that this is only against malicious Alice. We get the following bound without restricting which party is adversarial:

$$\Pr_{\mathsf{trunc}(\pi,i)}[\mathcal{R}|xy] \leq (52|Y|c)\,\varepsilon_i = \varepsilon_{i+1}$$

Truncating directly before $\mathcal{R}$, then, will only increase the simulation error to ε_{i+1}.

Lemma (Restatement of Lemma 16). *If, for incomplete and non-unilateral f and g, $f \sqsubseteq g$ using a protocol with security parameter κ and strict upper bound on number of rounds $r = O(\log \kappa)$, then f embeds in g.*

Proof. $\mathsf{trunc}(\pi, 0)$ has simulation error ε which is surely less than ε_0.

Either f embeds in g or we can apply the argument in Lemma 24 up to $r-1$ times. If $r = O(\log \kappa)$, then we are left with a 1-round protocol $\mathsf{trunc}(\pi, r-1)$ with simulation error $\varepsilon_r = (52nc)^{O(\log \kappa)}\sqrt{\varepsilon} = \mathsf{poly}(\kappa)\kappa^{-\omega(1)} = \kappa^{-\omega(1)}$ which is negligible. Then $f \sqsubseteq_1 g$, which by Lemma 6 means that f actually does embed in g.

E Round Complexity & Parallel Calls to g

Our model encompasses protocols that make only a single call to g in each round. Requiring sequential calls to g is without loss of generality *with respect to security*, since in the UC model it cannot be *guaranteed* that calls happen in parallel. The adversary can without loss of generality schedule all the calls sequentially (learning the output of one before choosing an input to the next), resulting in a protocol in our model.

However, when considering round complexity, it is more realistic to allow protocols that make *parallel calls* to g. Although the adversary can schedule these parallel calls in sequence, we still consider the round complexity as that required by the *honest* parties. Most of our technical results apply to such protocols. More formally, consider protocols where at each step the parties may call n parallel instances of g, where n is agreed-upon by both parties. All of our results in Section 5.3 and most of the results in Section 6 apply to such protocols. In particular, we can collapse any $f \sqsubseteq g$ protocol of $O(\log \kappa)$ rounds to a single-round protocol that may make *many* parallel calls to g but uses no additional communication.

To extend our results, it suffices to show that such a protocol (many parallel calls to g, no additional communication) implies that f embeds in g. We have been currently unable to extend this step. The way we currently extract an embedding from a *single-call* protocol (Lemma 6) works by derandomizing the protocol, crucially using the fact that the number of possible actions in the protocol is *constant*. This property is not true when a protocol makes, say, $O(\kappa)$ parallel calls to g.[7]

[7] Our results hold as stated for protocols that call at most $O(\log \kappa)$ instances of g in parallel at a time, where the number of possible actions is polynomial in κ.

In Section 7.2 we gave a $\omega(\log \kappa)$-round protocol for specific f using specific g. We point out that this particular protocol *cannot* be made constant-round by making all the g-calls in parallel. The simple attack is to split the g-calls into two groups and use different effective inputs in both (e.g., Alice uses inputs from the first row-group in half of the calls, and second row-group in the other half). With very good probability, this attack leaks as much as evaluating f on two inputs. In particular, Alice can learn Bob's input in its entirety from this attack.

We conjecture that there is no $O(\log \kappa)$-round protocol for this f using this g, even when the protocol allows unlimited parallel calls to g in each round.