Review of the book

"Bijective Combinatorics"

Nicholas A. Loehr

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Dr Kian B. Tay
Center for Strategic Infocomm Technologies

1 What the book is about

This very recent book (590 pages) aims to be an introductory text on bijective combinatorics (using bijections to solving counting problems). It is more suitable as a second course on combinatorics as the problems considered in this book requires more techniques in algebra. It has enough materials for a full year course.

Bijective combinatorics is the study of basic principles of enumerative combinatorics with emphasis on the role of bijective proofs. Enumerative combinatorics by itself is the mathematical theory of counting. How many functions map a 10 element set onto a 7 element set? How many ways can we divide an assembly of 20 people into 5 groups? How many ways can we write a positive integer n as a sum of positive integers? The techniques of enumerative combinatorics allow us to find answers to question like these.

This text contains a systematic development of the mathematical tools that are needed to solve these enumeration problems: basic counting rules, recursions, inclusion-exclusion techniques, generating functions, bijective proofs and linear algebraic methods. These tools are used to analyse many combinatorial structures such as words, permutations, subsets, integer partitions, graphs, trees, lattice paths, multisets, rook placements, set partitions, derangements, posets, tilings and abaci. It also delves into algebraic tools of combinatorics like how to solve problems using symmetric groups, group actions, symmetric polynomials, determinants and tableau theory.

The author began his book by teasing the readers with 3 counting problems:

1. Standard tableau
2. Rook placements
3. Tilings

Standard Tableau: A diagram D consisting of a number of rows of n boxes, left justified, with each row no longer than the one above it. A standard tableau of shape D is a filling of the boxes in D with the integers 1,2,..., n (used exactly once each) so that every row forms an increasing sequence (left to right), and every column forms an increasing sequence from top to bottom.

Question: Given a diagram D of n cells, how many standard tableaux of shape D are there?

Rook Placements: A rook is a chess piece that can travel any number of squares along its current row or column in a single move.We say that the rook attacks all the squares in its row and columns.

Question: Consider an (n+1) x (n+1) chessboard with a bishop occupying the upper-left corner. How many ways can we place n rooks on this chessboard so that no 2 pieces attack one another?

Tilings: A domino is a rectangular object that can cover 2 horizontal or vertical adjacent squares on a chessboard. A tiling of a board is a covering of the board with dominos such that each square is covered by exactly one domino.

Question: Given an m x n board, how many ways can we tile it with dominos?

All these 3 problems have nice explicit answers and the proofs of which needed the tools of algebraic combinatorics.

For the standard tableau problem, it turns out that the concept of hook-length of each box in the tableau is crucial. In fact the number of standard tableaux of shape D is the number n! divided by the product of all the hook lengths of all the boxes in the tableau.

For rook placements, the number of ways we can arrange the rooks is n! times the alternating sum of 1/k! (n terms), which can be shown to be equal to [n!/e], the closest integer to n!/e.

For the tiling problem, the required number of tilings can be written explicitly as a double product of terms, where the terms are squareroots of sums of cosine squares!

How to prove all these? These proofs are all covered in the text, with the various algebraic tools introduced along the way.

2 What the book is like

There are the **12 chapters** of this book:

Chapter 1: Basic Counting (33 pages)
Besides the basic standard topics in counting, this chapter also covers lattice paths which turns out to be useful in realising proofs of some combinatorial identities, the most famous of which is the binomial identity. Dyck paths and Catalan numbers are first introduced in this chapter. Multisets are also covered, along with probability, including conditional probability.

Chapter 2: Combinatorial Identities and Recursions (49 pages)
Recursions, more combinatorial identities, integer and set partitions are covered here, along with Stirling numbers and its relations to rook theory.

Chapter 3: Counting Problems in Graph Theory (47 pages)
Chapter 3 deals with famous counting problems that arise in graph theory, namely enumerating tress, colourings of graphs, chromatic polynomials and Eulerian tours.

Chapter 4: Inclusion-exclusion and Related Techniques (23 pages)
Deals with inclusion and exclusion principles and applications to counting derangements, Euler's phi function. Classical Mobius inversions, including inversions for partially ordered sets (posets) are dealt here.

Chapter 5: Ranking and Unranking (26 pages)
This chapter studies the notions of ranking and unranking from a bijective viewpoint.

Chapter 6: Counting Weighted Objects (25 pages)
This chapter generalises the counting problem to counting of weighted objects: Given a finite set of objects, each object is assigned an integer weight, how many objects are there of each given weight?

Chapter 7: Formal Power Series (42 pages)
Chapter 8: The Combinatorics of Formal Power Series (23 pages)
Chapter 7 and 8 deal with formal power series and the combinatorics of it, with applications to counting trees, integer and set partitions.

Chapter 9: Permutations and group actions (49 pages)
This chapter contains aspects of group theory that are related to some counting problems. Basic group theory is covered, with emphasis on symmetric groups. The author applies this to the calculation of determinants. He also covers group actions which has many applications to counting

problems involving symmetries, like the colouring of square chessboard with a certain number of colours, discounting reflections and rotations.

Chapter 10: Tableaux and Symmetric Polynomials (60 pages)
The author studied the theory of tableau here, which he uses to define combinatorically the famous Schur polynomials, which are special classes of symmetric polynomials. We will see the theory of symmetric polynomials nicely demonstrates the interplay between combinatorics and algebra.

Chapter 11: Abaci and Antisymmetric polynomials (49 pages)
This chapters investigates the interplay between the combinatorics of abaci and the algebraic properties of antisymmetric polynomials.

Chapter 12: Additional topics (51 pages)
Having covered all the various tools, the author now has the freedom to cover many interesting topics, like the famous Chung-Feller theorem, rook equivalence of Ferrer boards, parking functions, tournaments, hook-length formula which is used in the counting of standard tableau of given shape are covered here. We can also find Pfaffians and perfect matchings, and they are used in the calculation of the total number of domino tilings of m x n rectangles.

3 What I like about this book

It provides a rigorous exposition of many topics in algebraic combinatorics. Proofs involving bijections are emphasized and this links up seemingly unrelated topics. It requires less mathematical pre-requisites as “outside” materials (like group theory) are covered. It covers important topics like symmetric polynomials and group actions from a combinatorial viewpoint, all in a single book. It contains numerous worked examples and applications as well as nearly 1000 exercises ranging from the elementary to research level problems, with hints and solutions to many of the exercises. The author also provides an overview at the beginning of each chapter and he ends each chapter with a summary of the key results.

4 Possible Improvements

Brief answers to every less trivial problem in the book will be helpful to those who want to check if they have done the problem correctly. Challenging problems should be marked with a * so that students will not be unduly discouraged if they cannot find the answer. More detailed remarks at the end of each chapter to briefly say about new developments in that area (with references) would be very helpful. Many of the authors' arguments are tightly packed in a single page and often in long paragraphs. Split the longer paragraphs into shorter ones (eg page 391, 486). For longer proofs, it will be good if he can split into many small parts, and tell the reader the gist of what he is about to do. Highlight the key ideas in each part before the whole long proof is given.

5 Would you recommend this book?

I like this book because it covers topics that I had longed to learn. I could follow the arguments in the book (my doctorate is in pure math, not cryptography). However I feel that this book is geared towards people who are bright math or CS seniors to graduate students.
Those who are not as bright mathematically will need a teacher/tutor to guide him or her through. The book might seem hard to plough through at first, but if one is patient, one can work through the book and gain great insights into the fertile field of algebraic combinatorics.

The reviewer is a researcher in infocomm security with specialty in math and cryptography. He was formerly a professor in math.