Quantum Key Recycling

Joint work with
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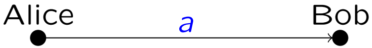
Outline

Motivation and Previous Work



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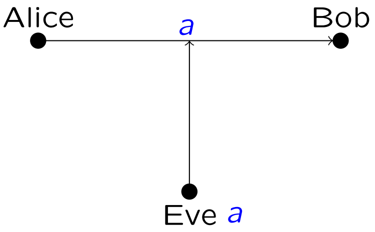

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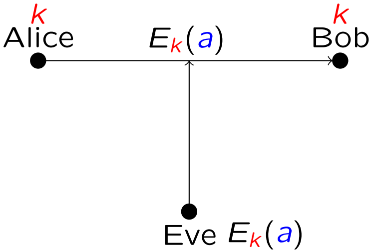

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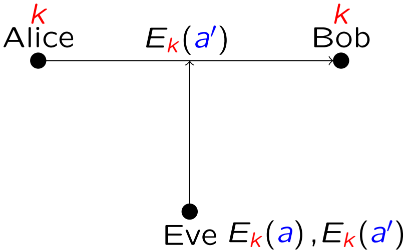

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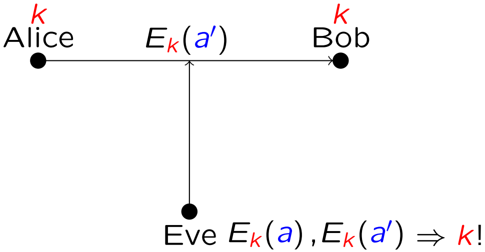

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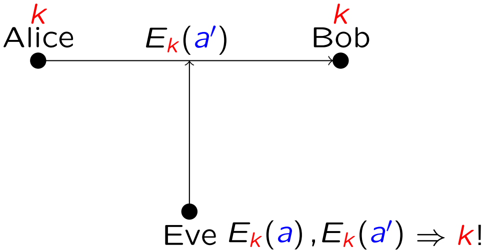

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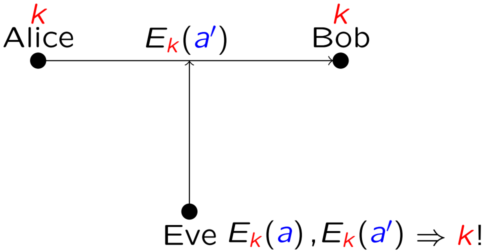

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- Is there a way to detect eavesdropping?

Quantum Authentication

- Detects eavesdropping.

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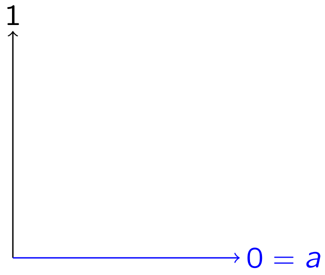

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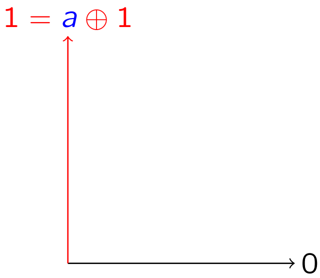

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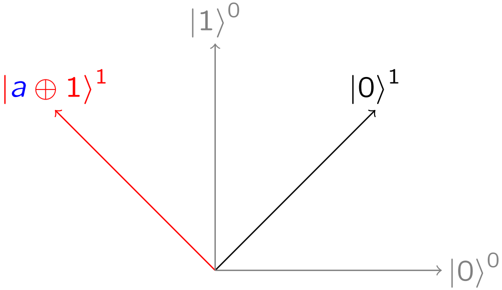

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- Change quantum representation to basis 1.

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Theorem

$H(K|a, \mathcal{E}_{(z,b)}(a)) \geq 2n - 1$ *(EUROCRYPT 04)*.

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- Bob decrypts, verifies authentication, and send $\mathrm{acc}/\mathrm{rej}$ to Alice.
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- If Bob rejects z is replaced.

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Theorem

Under eavesdropping $t \leq n - m + 1$ *bits are recycled.*

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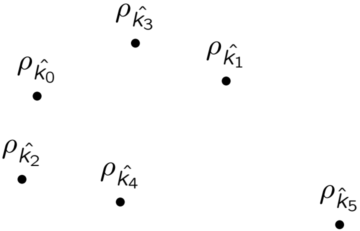

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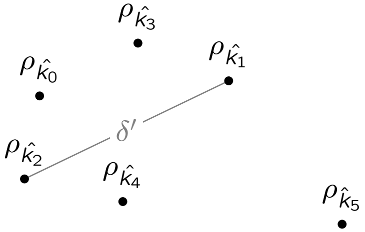

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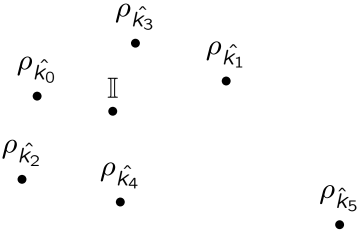

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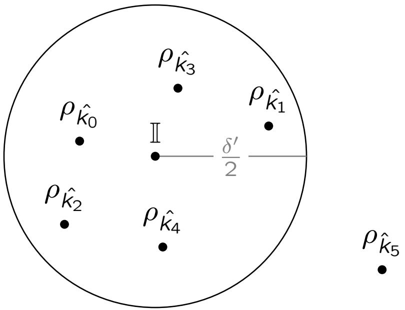

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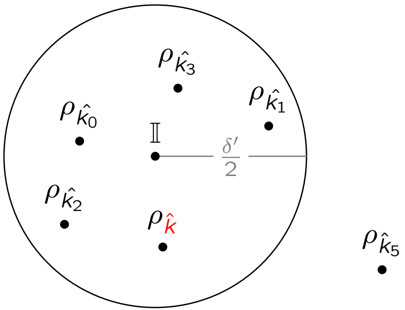

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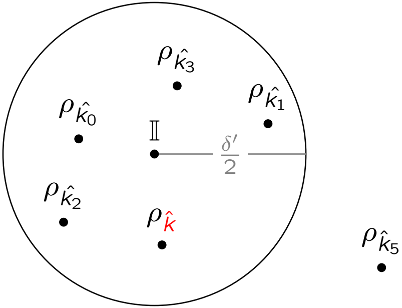

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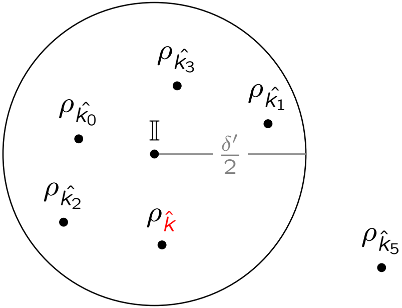

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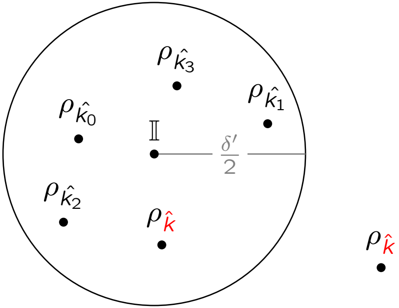

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- The result follows by contradiction.

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- $(0 \leq |c_1|^2 \leq 1)$ probability of no eavesdropping.
- $p_{\mathrm{acc}} \leq |c_1|^2 + \mathrm{negligible}(n)$.

Conclusion

- Detecting eavesdropping.
- Worst case quantum = classical.
- Best case: entire key can be reused.