SECRET SHARING SCHEMES FOR VERY DENSE GRAPHS

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Program

1. Introduction to Secret Sharing
2. Graph Secret Sharing
3. Our Results

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A method to protect a secret

Secret Sharing Scheme

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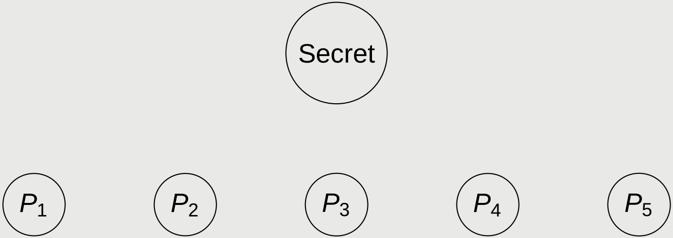

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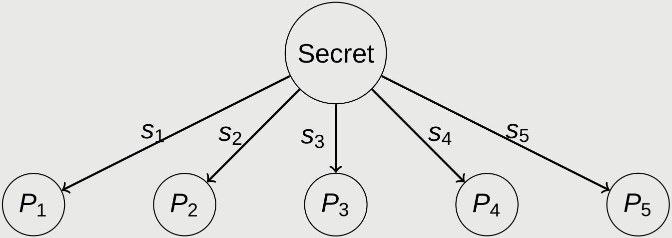

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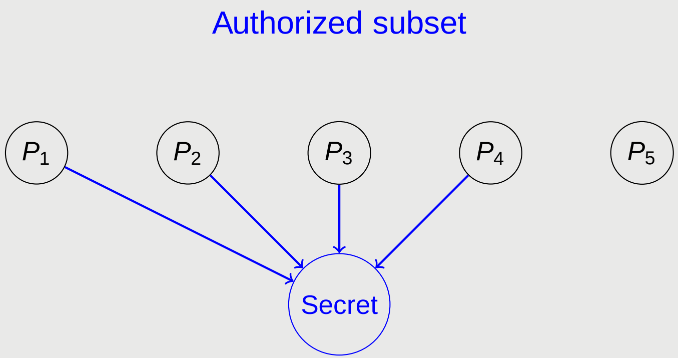

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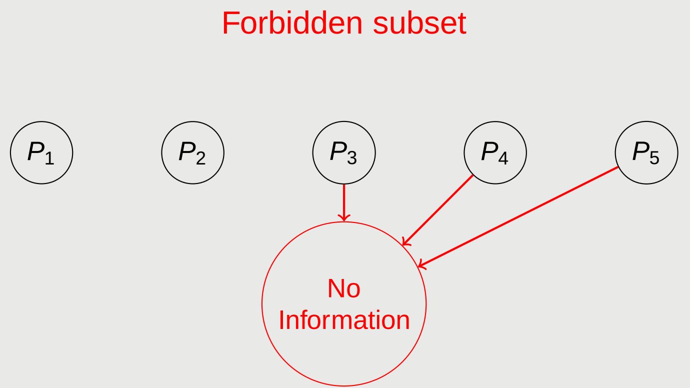

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Access structure: Family of authorized subsets

P_1 P_2 P_3 P_4 P_5

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- Access control
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Need of efficient schemes: Shares have to be small.

On the Share Size

As a measure of efficiency we use

$$\textcolor{red}{\text{total share size}}\text{:}\ \frac{\text{sum size of shares}}{\text{size of secret}} = \frac{\sum \log |S_i|}{\log |S|}$$

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For most access structures, the t.s.s of these schemes is $2^{O(n)}$.

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There is a family of access structures for which the t.s.s. of any scheme is

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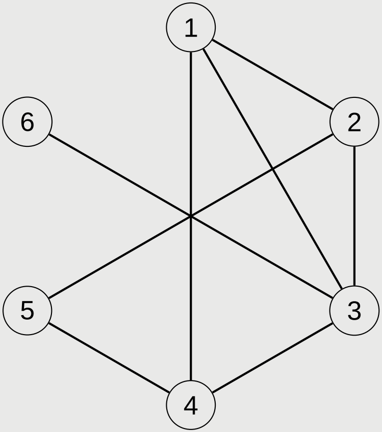
We extend the techniques for finding upper bounds to

- homogeneous access structures
- the general case

Graph Secret Sharing



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It defines a graph.

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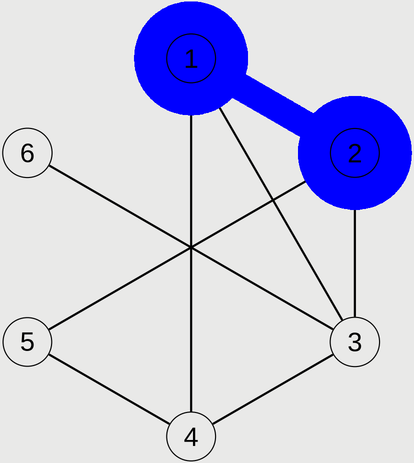

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A set is authorized if contains an edge:

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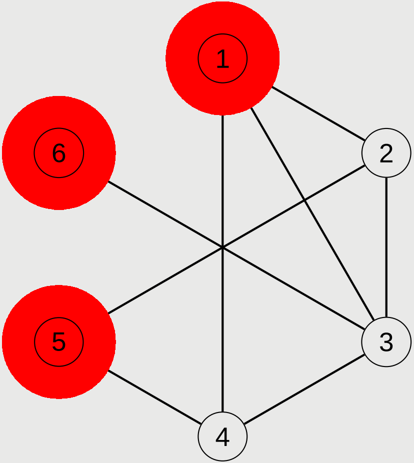

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A set is forbidden if does not contain any edge:

$$\{1, 5, 6\}$$

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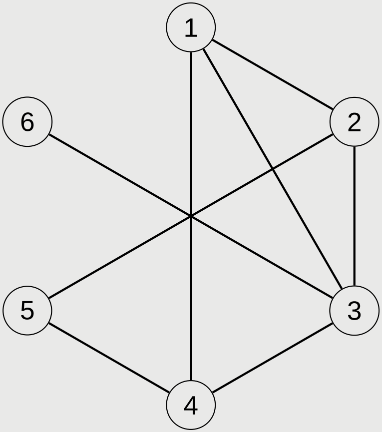

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A graph secret sharing scheme is a scheme with graph access structure.

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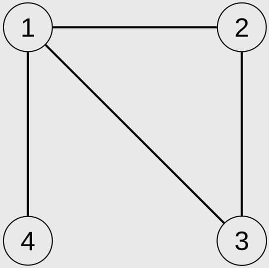

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The secret s is shared independently for every edge.

The total share size is $2m$, where m is the number of edges.

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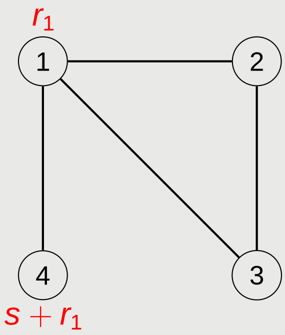

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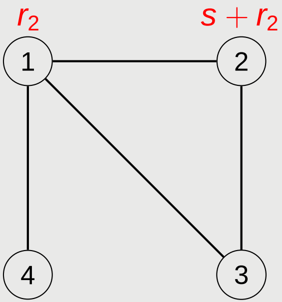

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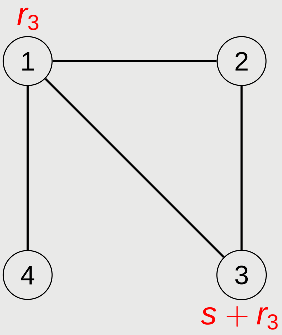

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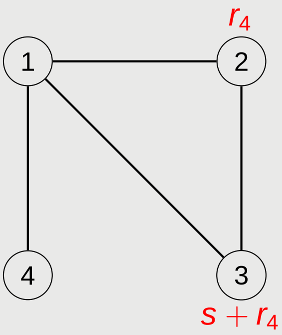

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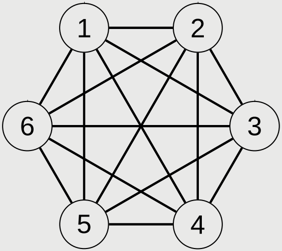

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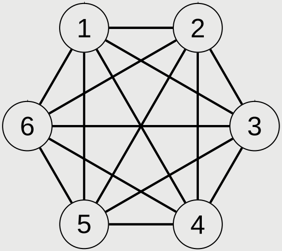

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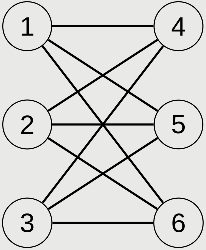
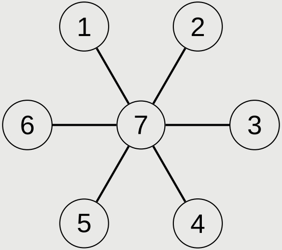
Complete Bipartite Graph



Star



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- i.e. graphs with $\binom{n}{2} - \ell$ edges, with ℓ "small".

Our Main Result

Theorem

If a graph has $\binom{n}{2} - n^{1+\beta}$ edges for some $0 < \beta < 1$, then it admits a scheme with total share size

$$O(n^{5/4+3\beta/4} \log n).$$

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If a graph has $\binom{n}{2} - n^{1+\beta}$ *edges for some* $0 < \beta < 1$*, then it admits a scheme with total share size*

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Main techniques:

- Coverings by "easy" graphs: cliques, bipartite graphs and stars.
- The probabilistic method.
- Colorings of graphs. skip details

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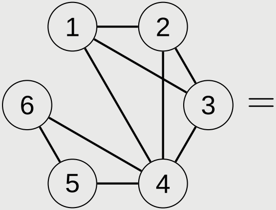

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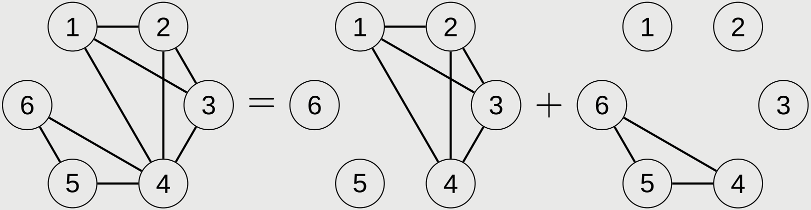

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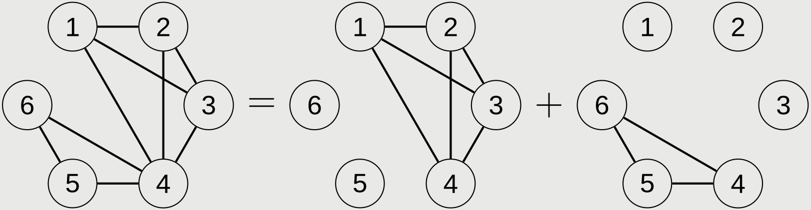

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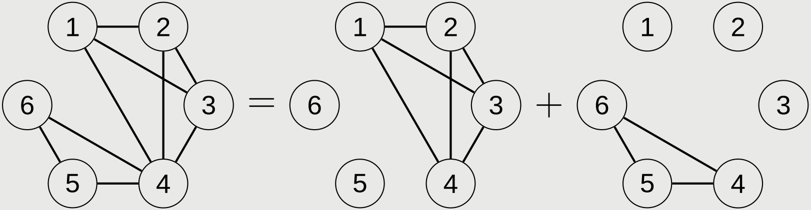

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We look for small coverings in order to obtain efficient schemes.

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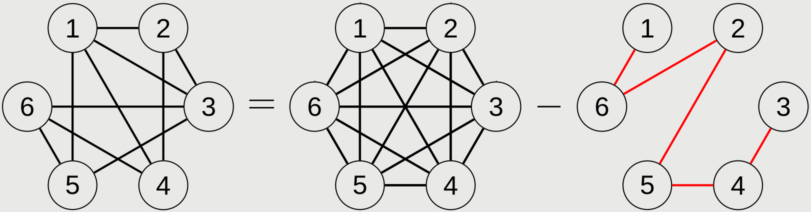

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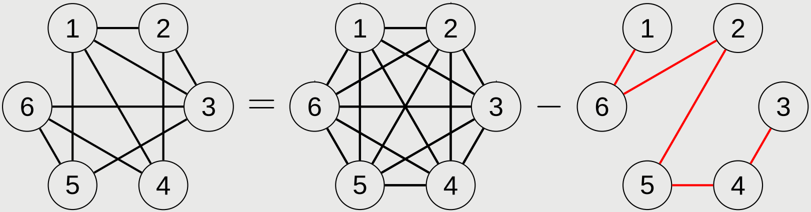

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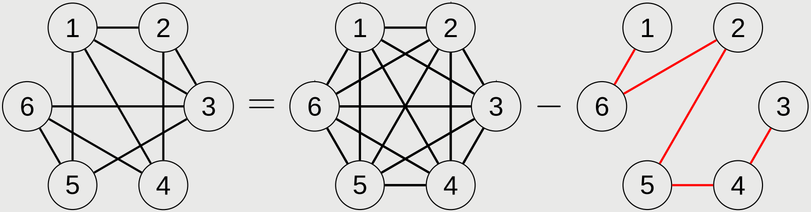

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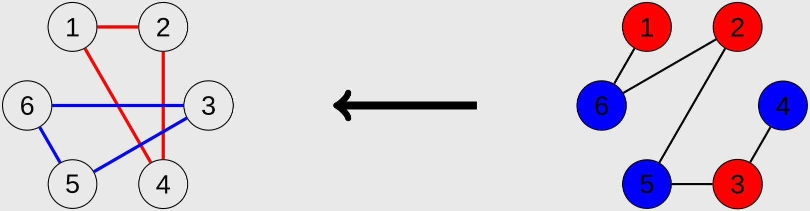

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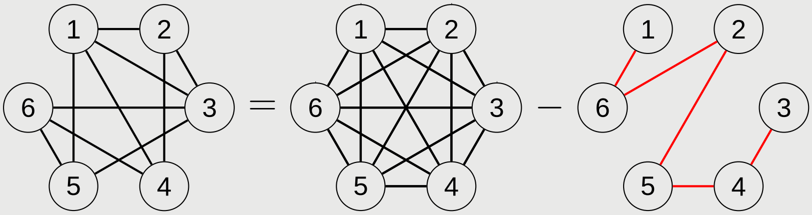

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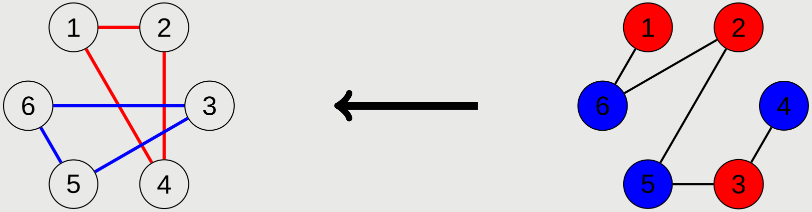

Taking many random colorings of G' we end with a covering of G.

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If a graph has $\binom{n}{2} - n^{1+\beta}$ edges for some $0 < \beta < 1$, then it admits a scheme with total share size

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If a graph has $\binom{n}{2} - \ell$ edges,

- *and $\ell < n/2$, then it admits a scheme with t.s.s. $n + O(\ell^{5/4})$.*
- *and $\ell \ll n^{4/5}$, then it admits a scheme with t.s.s. $n + o(n)$.*

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Direct consequence: If G admits an efficient scheme, the graphs that are close to G are not hard.

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go to details

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We provide new techniques and constructions, and we give answers to the following problems:

- Deleting minimal authorized subsets in a threshold access structure.
- Deleting minimal authorized subsets in any access structure.

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There exists a scheme with t.s.s. $\tilde{O}(n^{5/4+3\beta/4}) = \tilde{O}(n^{1+\beta/2} \cdot n^{1/4+\beta/4})$.

Summary and Open Directions

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- Secret sharing for very dense graphs.
- New upper and lower bounds for the total share size for the schemes realizing these graphs.
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Open directions:

- To find hard graphs.
- New techniques for finding lower bounds on the total share size.
- To bridge the gap between upper and lower bounds on the total share size.

Thank you

Appendix: Deleting Minimal Authorized Subsets

Theorem

Let Γ be a t-threshold access structures for a constant t. If we delete ℓ minimal authorized subsets, then the resulting access structure admits a scheme with total share size

$$\tilde{O}(\ell n).$$

Let Γ be an access structure with a scheme of t.s.s. r such that

- if $A \in \min \Gamma$, then $|A| \leq k$ for some constant k.

Let Γ' be an access structure such that

- $\min \Gamma' = \min \Gamma \setminus \Delta$
- for every $p \in P$, there is at most d subsets in Δ containing p.

Theorem

The access structure Γ' admits a scheme with total share size

$$\tilde{O}(d^{k-1} r).$$