Pseudorandom Functions and Lattices

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Outline

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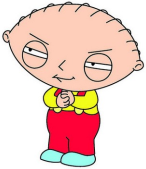
Pseudorandom Functions [GGM'84]

- A family of functions $\mathcal{F} = \{F_s : \{0,1\}^k \to B\}$ such that, given adaptive query access,

$$\boxed{F_s \leftarrow \mathcal{F}} \quad \overset{c}{\approx} \quad \boxed{\text{Random } U}$$

$x_i \uparrow \quad \downarrow F_s(x_i) \qquad\qquad x_i \uparrow \quad \downarrow U(x_i)$



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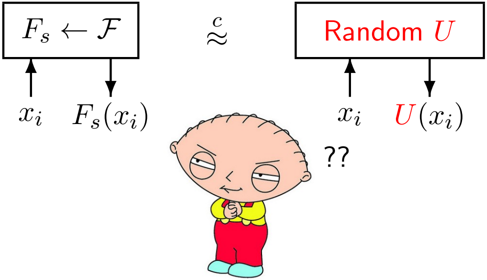

- Lots of applications in symmetric key cryptography: encryption, message authentication, friend or foe identification...

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Cooking a PRF

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    - Based on any (doubling) PRG. $F_s(x_1, \ldots, x_k) = G_{x_k}(\ldots(G_{x_1}(s))\ldots)$

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3. Direct constructions [NR'95, NR'97, NRR'00]
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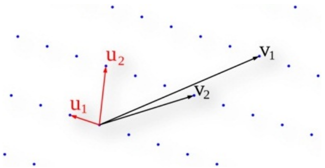
3. Direct constructions [NR'95, NR'97, NRR'00]
   - Parallel and theoretically efficient
   - Security based on number-theory (DDH, factoring...)
   - Not practically efficient (huge exponentiations), lots of preprocessing
   - What about a "post-quantum" world?

Lattices



A periodic grid in the n-dimensional Euclidean space

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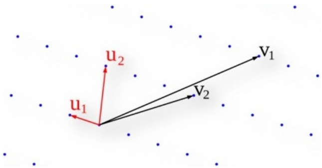

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Advantages of Lattice Crypto Schemes

- Simple & efficient: linear, highly parallel operations
- Resist quantum attacks (so far)
- Secure under worst-case hardness assumptions [Ajt'96,. . .]

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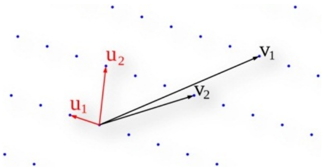

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Lattice-based Pseudorandomness?

- Only known PRF is generic GGM, no direct constructions
- Security proofs based on hard lattice problems need fresh biased errors

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Full version: `http://eprint.iacr.org/2011/401`

Outline

(Ring-) Learning with Errors: (R)LWE [Reg'05,Pei'09,LPR'10]

- For n a power of 2 (say), define "cyclotomic" polynomial rings

$$R := \mathbb{Z}[x]/(x^n + 1) \quad \text{and} \quad R_q := R/qR = \mathbb{Z}_q[x]/(x^n + 1)$$

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- (R)LWE Problem:

$$\boxed{(a_i, a_i \cdot s + e_i)} \overset{c}{\approx} \boxed{(a_i, b_i) \in R_q \times R_q}$$

$s \in R_q$ uniform and fixed

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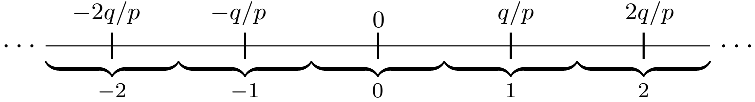
- Secret errors e_i need fresh randomness. Can we make this deterministic?

"Learning with Rounding": LWR [This Work]

- Generate errors deterministically by rounding to a "sparse" subgroup (Fundamental operation used in decryption algorithms)

  Let $p < q$ and define $\lfloor x \rceil_p = \lfloor (p/q) \cdot x \rceil \bmod p$ for $x \in \mathbb{Z}_q$



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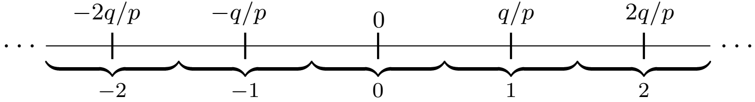

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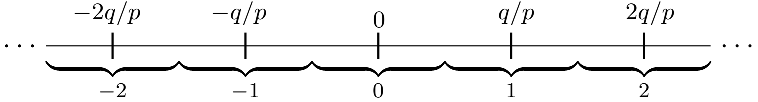

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- LWE conceals low-order bits by adding small random noise
- LWR discards low-order bits instead

Security Proof and Application

Theorem

Ring-LWE$_{q,\chi} \leq$ *RLWR*$_{q,p}$ *when* $q \geq p \cdot n^{\omega(1)}$ *and* χ *short*

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Synthesizers from LWR

- $S \colon R_q \times R_q \to R_p$ defined as $S(a, s) = \lfloor a \cdot s \rceil_p$ is a synthesizer [NR'95]
- Gives a k-bit PRF through a $\log k$ depth tree of synthesizers
- Details of the construction in the paper

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1. Introduction
2. Learning with Rounding
3. Direct Construction

Direct Construction: Shallower? More Efficient?

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- [NR'97,NRR'00]: direct PRFs from DDH / factoring (in NC^1)

$$F_{g,s_1,\ldots,s_k}(x_1 \cdots x_k) = g^{\prod s_i^{x_i}}$$

(Computing this needs a costly exponentiation or lots of preprocessing...)

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  $\Rightarrow$ New function $F'(x) = \lfloor a_{x_1} \cdot s_2^{x_2} \cdots s_k^{x_k} \rceil_p$

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- Repeat inductively for $s_2, s_3, \ldots$ until we get the Uniform func

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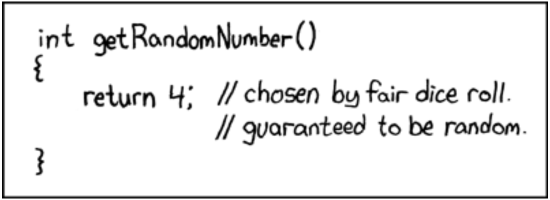
Open Questions

1. Get different proofs with better p/q ratios:

	LWR	Synth-PRF	Direct-PRF
Ratio (Current)	$n^{\omega(1)}$	$n^{\Theta(\log k)}$	$n^{\Theta(k)}$
Ratio (Hope)	$\sqrt{n}$	$\mathrm{poly}(n)$	$\mathrm{poly}(n)$

2. Efficient PRF from parity with noise (LPN) or subset sum?

The Last Word [Mun'07]



(Image source: `http://xkcd.com/221/`)